Introduction to Time Series Analysis. Lecture 15.

Spectral Analysis

1. Spectral density: Facts and examples.
2. Spectral distribution function.
3. Wold's decomposition.

Spectral Analysis

Idea: decompose a stationary time series $\{X_t\}$ into a combination of sinusoids, with random (and uncorrelated) coefficients.

Just as in Fourier analysis, where we decompose (deterministic) functions into combinations of sinusoids.

This is referred to as 'spectral analysis' or analysis in the 'frequency domain,' in contrast to the time domain approach we have considered so far.

The frequency domain approach considers regression on sinusoids; the time domain approach considers regression on past values of the time series.

A periodic time series

Consider

$$
\begin{aligned}
X_t &= A\sin(2\pi\nu t) + B\cos(2\pi\nu t) \\
&= C\sin(2\pi\nu t + \phi),
\end{aligned}
$$

where A, B are uncorrelated, mean zero, variance $\sigma^2 = 1$, and $C^2 = A^2 + B^2$, $\tan\phi = B/A$. Then

$$
\begin{aligned}
\mu_t &= \mathrm{E}[X_t] = 0 \\
\gamma(t, t+h) &= \cos(2\pi\nu h).
\end{aligned}
$$

So $\{X_t\}$ is stationary.

An aside: Some trigonometric identities

$$
\begin{aligned}
\tan\theta &= \frac{\sin\theta}{\cos\theta}, \\
\sin^2\theta + \cos^2\theta &= 1, \\
\sin(a+b) &= \sin a\cos b + \cos a\sin b, \\
\cos(a+b) &= \cos a\cos b - \sin a\sin b.
\end{aligned}
$$

A periodic time series

For $X_t = A\sin(2\pi\nu t) + B\cos(2\pi\nu t)$, with uncorrelated A, B (mean 0, variance σ^2), $\gamma(h) = \sigma^2\cos(2\pi\nu h)$.

The autocovariance of the sum of two uncorrelated time series is the sum of their autocovariances. Thus, the autocovariance of a sum of random sinusoids is a sum of sinusoids with the corresponding frequencies:

$$X_t = \sum_{j=1}^{k}\left(A_j\sin(2\pi\nu_j t) + B_j\cos(2\pi\nu_j t)\right),$$

$$\gamma(h) = \sum_{j=1}^{k}\sigma_j^2\cos(2\pi\nu_j h),$$

where A_j, B_j are uncorrelated, mean zero, and $\mathrm{Var}(A_j) = \mathrm{Var}(B_j) = \sigma_j^2$.

A periodic time series

$$X_t = \sum_{j=1}^{k} \left(A_j \sin(2\pi\nu_j t) + B_j \cos(2\pi\nu_j t) \right), \quad \gamma(h) = \sum_{j=1}^{k} \sigma_j^2 \cos(2\pi\nu_j h).$$

Thus, we can represent $\gamma(h)$ using a Fourier series. The coefficients are the variances of the sinusoidal components.

The *spectral density* is the continuous analog: the Fourier transform of γ.

(The analogous *spectral representation* of a stationary process X_t involves a *stochastic integral*—a sum of discrete components at a finite number of frequencies is a special case. We won't consider this representation in this course.)

Spectral density

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its **spectral density** as

$$f(\nu) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h}$$

for $-\infty < \nu < \infty$.

Spectral density: Some facts

1. We have $\sum_{h=-\infty}^{\infty} \left|\gamma(h)e^{-2\pi i\nu h}\right| < \infty$.
   This is because $|e^{i\theta}| = |\cos\theta + i\sin\theta| = (\cos^2\theta + \sin^2\theta)^{1/2} = 1$, and because of the absolute summability of γ.

2. f is periodic, with period 1.
   This is true since $e^{-2\pi i\nu h}$ is a periodic function of ν with period 1.

   Thus, we can restrict the domain of f to $-1/2 \leq \nu \leq 1/2$. (The text does this.)

Spectral density: Some facts

3. f is even (that is, $f(\nu) = f(-\nu)$).
   To see this, write

$$f(\nu) = \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu h} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu h},$$

$$\begin{aligned} f(-\nu) &= \sum_{h=-\infty}^{-1} \gamma(h) e^{-2\pi i \nu(-h)} + \gamma(0) + \sum_{h=1}^{\infty} \gamma(h) e^{-2\pi i \nu(-h)}, \\ &= \sum_{h=1}^{\infty} \gamma(-h) e^{-2\pi i \nu h} + \gamma(0) + \sum_{h=-\infty}^{-1} \gamma(-h) e^{-2\pi i \nu h} \\ &= f(\nu). \end{aligned}$$

4. $f(\nu) \geq 0$.

Spectral density: Some facts

5. $\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu.$

$$\begin{aligned}
\int_{-1/2}^{1/2} e^{2\pi i\nu h} f(\nu)\, d\nu &= \int_{-1/2}^{1/2} \sum_{j=-\infty}^{\infty} e^{-2\pi i\nu(j-h)} \gamma(j)\, d\nu \\
&= \sum_{j=-\infty}^{\infty} \gamma(j) \int_{-1/2}^{1/2} e^{-2\pi i\nu(j-h)}\, d\nu \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)}{2\pi i(j-h)} \left(e^{\pi i(j-h)} - e^{-\pi i(j-h)} \right) \\
&= \gamma(h) + \sum_{j\neq h} \frac{\gamma(j)\sin(\pi(j-h))}{\pi(j-h)} = \gamma(h).
\end{aligned}$$

Example: White noise

For white noise $\{W_t\}$, we have seen that $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.

Thus,

$$\begin{aligned} f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h} \\ &= \gamma(0) = \sigma_w^2. \end{aligned}$$

That is, the spectral density is constant across all frequencies: each frequency in the spectrum contributes equally to the variance. This is the origin of the name *white noise*: it is like white light, which is a uniform mixture of all frequencies in the visible spectrum.

Example: AR(1)

For $X_t = \phi_1 X_{t-1} + W_t$, we have seen that $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1-\phi_1^2)$. Thus,

$$\begin{aligned}
f(\nu) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \nu h} = \frac{\sigma_w^2}{1-\phi_1^2} \sum_{h=-\infty}^{\infty} \phi_1^{|h|} e^{-2\pi i \nu h} \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left(1 + \sum_{h=1}^{\infty} \phi_1^h \left(e^{-2\pi i \nu h} + e^{2\pi i \nu h} \right) \right) \\
&= \frac{\sigma_w^2}{1-\phi_1^2} \left(1 + \frac{\phi_1 e^{-2\pi i \nu}}{1-\phi_1 e^{-2\pi i \nu}} + \frac{\phi_1 e^{2\pi i \nu}}{1-\phi_1 e^{2\pi i \nu}} \right) \\
&= \frac{\sigma_w^2}{(1-\phi_1^2)} \frac{1-\phi_1 e^{-2\pi i \nu} \phi_1 e^{2\pi i \nu}}{(1-\phi_1 e^{-2\pi i \nu})(1-\phi_1 e^{2\pi i \nu})} \\
&= \frac{\sigma_w^2}{1-2\phi_1 \cos(2\pi\nu) + \phi_1^2}.
\end{aligned}$$

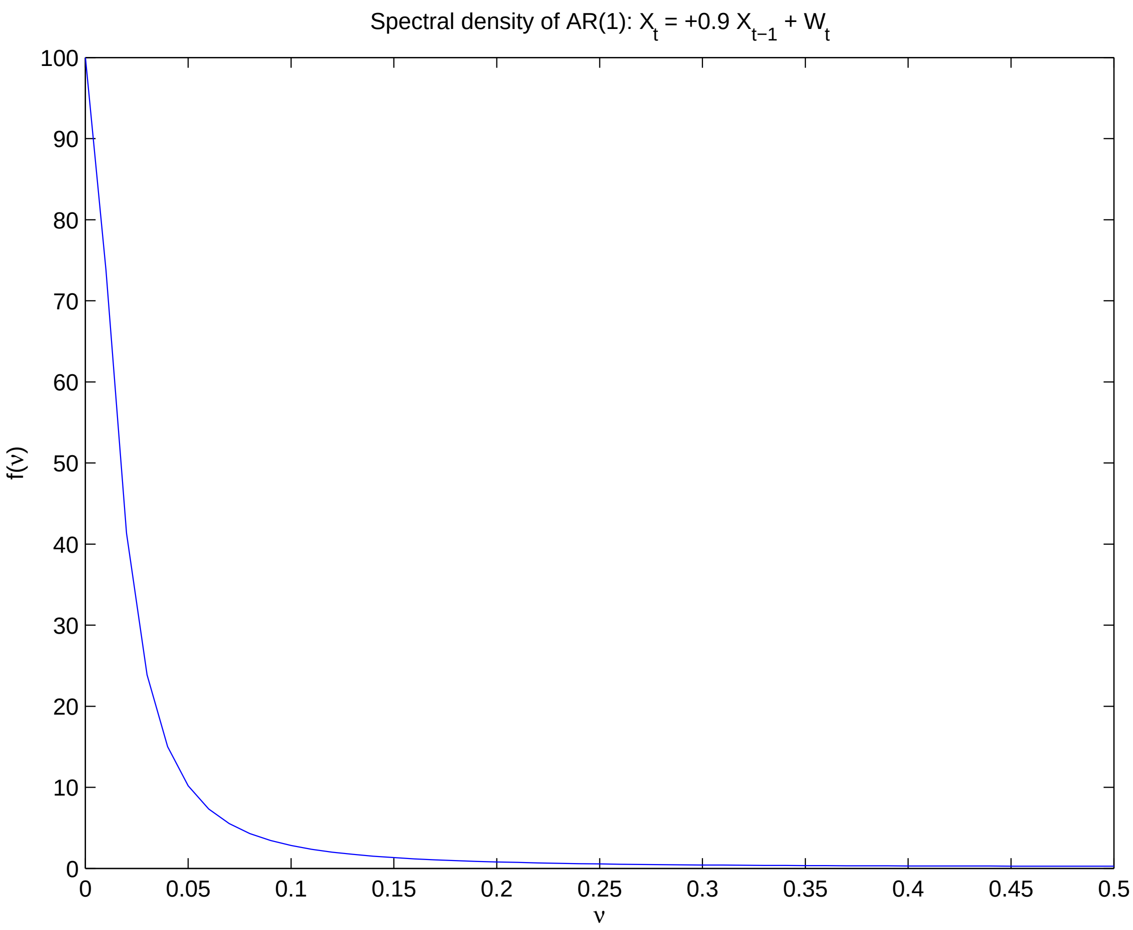
Examples

White noise: $\{W_t\}$, $\gamma(0) = \sigma_w^2$ and $\gamma(h) = 0$ for $h \neq 0$.
$f(\nu) = \gamma(0) = \sigma_w^2$.

AR(1): $X_t = \phi_1 X_{t-1} + W_t$, $\gamma(h) = \sigma_w^2 \phi_1^{|h|}/(1 - \phi_1^2)$.
$f(\nu) = \frac{\sigma_w^2}{1-2\phi_1 \cos(2\pi\nu)+\phi_1^2}$.

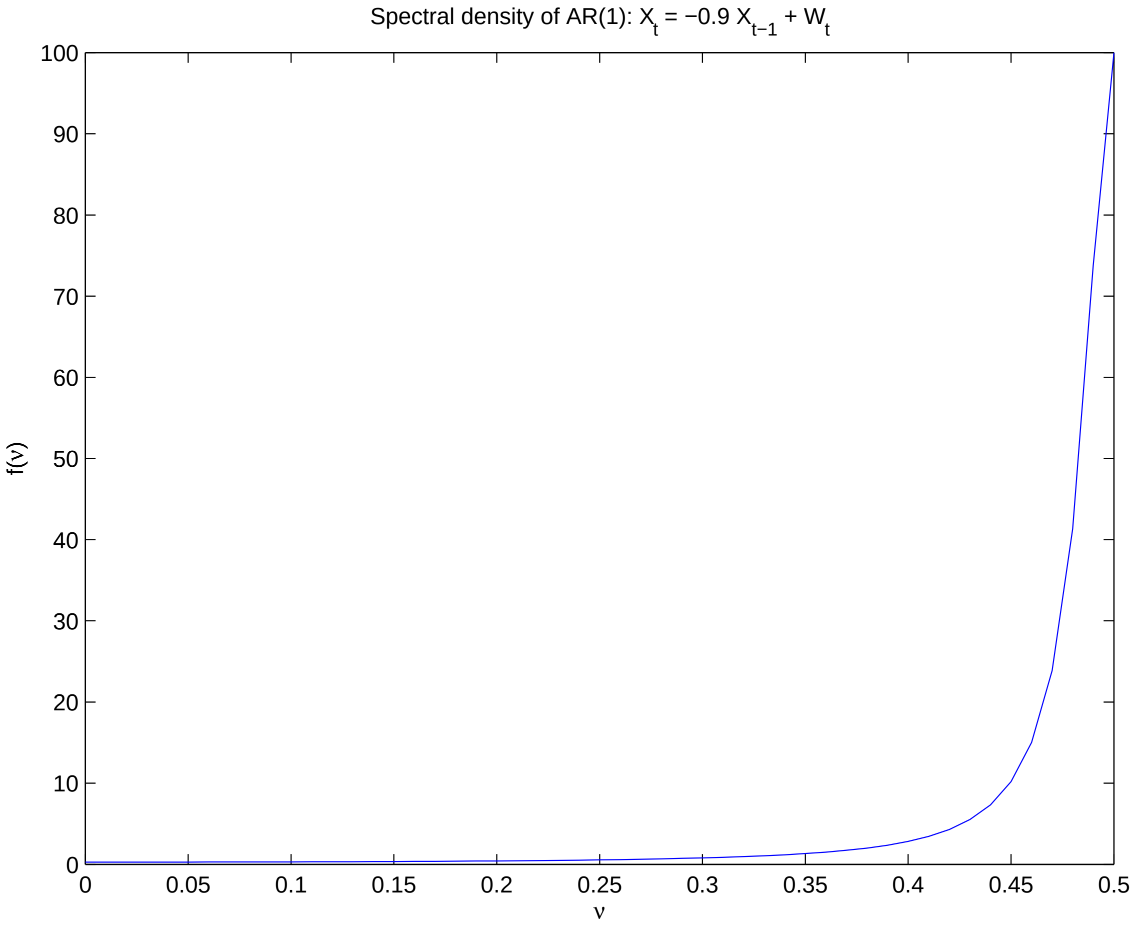
If $\phi_1 > 0$ (positive autocorrelation), spectrum is dominated by low frequency components—smooth in the time domain.
If $\phi_1 < 0$ (negative autocorrelation), spectrum is dominated by high frequency components—rough in the time domain.

Example: AR(1)



Spectral density of AR(1): X_t = +0.9 X_{t−1} + W_t
f(ν)
ν
100
90
80
70
60
50
40
30
20
10
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Example: AR(1)



Spectral density of AR(1): $X_t = -0.9\, X_{t-1} + W_t$
f(ν)
ν
0
10
20
30
40
50
60
70
80
90
100
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5