Homework 5 solutions

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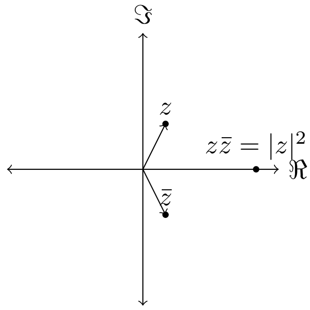
1. (a) Write $z = x+iy$. Then $z\bar{z} = (x+iy)(x-iy) = x^2+iyx-iyx-i^2y^2 = x^2+y^2 = |z|^2$.



(b) We prove this by induction: clearly $\overline{z^j} = \bar{z}^j$ when $j = 1$. Suppose $\overline{z^{j-1}} = \bar{z}^{j-1}$. Then

$$\overline{z^j} = \overline{zz^{j-1}} = \bar{z}\overline{z^{j-1}} = \bar{z}\bar{z}^{j-1} = \bar{z}^j.$$

(c) By part (b), $\overline{p(z)} = \sum_{j=1}^k \overline{a_j z^j} = \sum_{j=1}^j a_j \bar{z}^j = p(\bar{z})$. By part (a), $|p(z)|^2 = p(z)\overline{p(z)} = p(z)p(\bar{z})$.

2. (a) The MA and AR polynomials of X_t are $\theta(z) = 1-(4/5)^2z^2$ and $\phi(z) = 1-4\sqrt{2}/5z+(4/5)^2z^2)$. Therefore the spectral density is

$$f_X(\nu) = \left| \frac{1-\left(\frac{4}{5}\right)^2 e^{4\pi i\nu}}{1-4\frac{\sqrt{2}}{5}e^{2\pi i\nu}+\left(\frac{4}{5}\right)^2 e^{4\pi i\nu}} \right|^2.$$

The poles of $\theta(z)/\phi(z)$ occur at the zeros of ϕ (where $z = \frac{5\sqrt{2}}{8}(1\pm i)$) and the zeros of $\theta(z)/\phi(z)$ occur at the zeros of θ (where $z = \pm\frac{5}{4}$). The plot of these points is shown in Figure 1, from which we see that the zeros will cause the spectral density $f(\nu)$ to be small when $\nu = 0, \pm 1/2$ (so that $e^{2\pi i\nu}$ is close to $\pm 5/4$). The poles will cause the spectral density to be large when $\nu = \pm 1/8$ (so that $e^{2\pi i\nu}$ is close to $\frac{5\sqrt{2}}{8}(1\pm i)$).

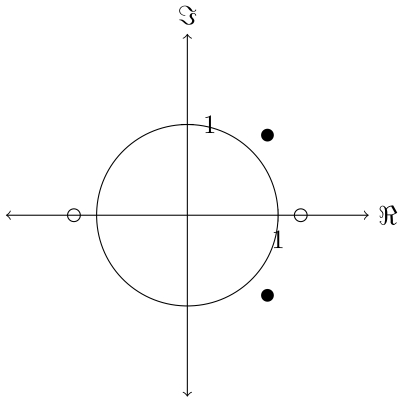


Figure 1: Zeros and poles of ψ_X (left) and ψ_Y (right). Poles are marked by filled circles; zeros are marked by unfilled circles. The unit circle is also drawn.

(b) We can write Y as a rational function of B acting on X:

$$Y_t = \frac{1}{1 - \frac{5}{6}B} X_t.$$

Therefore the spectral density of Y is

$$f_Y(\nu) = \left| \frac{1}{1 - \frac{5}{6}e^{2\pi i\nu}} \right|^2 f_X(\nu) = \left| \frac{1 - \left(\frac{4}{5}\right)^2 e^{4\pi i\nu}}{\left(1 - 4\frac{\sqrt{2}}{5}e^{2\pi i\nu} + \left(\frac{4}{5}\right)^2 e^{4\pi i\nu}\right)\left(1 - \frac{5}{6}e^{2\pi i\nu}\right)} \right|^2.$$

There is one extra pole in ψ_Y compared to ψ_X in the previous part (at $z = 6/5$). The plot of the zeros and poles is shown in Figure 1, from which we see that the extra pole is closer to the unit circle than the zero at 5/4. Therefore the spectral density of Y will be *large* at 0, while the spectral density of X will be small at 0. The spectral density of Y will still be small at $\pm 1/2$ and large at $\pm 1/8$.

3. The formula for ψ is

$$\psi(z) = \frac{1 - (\frac{4}{5})^2 z^2}{\left(1 - 4\frac{\sqrt{2}}{5}z + (\frac{4}{5})^2 z^2\right)\left(1 - \frac{5}{6}z\right)}.$$

The squared modulus of this function is plotted in Figure 2. The poles at $\frac{5\sqrt{2}}{8}(1 \pm i)$ and 6/5 are clearly visible. The zero at $-5/4$ is visible, but the zero at 5/4 is not, since it is hidden by the pole at 6/5.

4. (a) See Figure 3 for a plot of the periodograms. The code that generated them (and computed the confidence intervals) was:

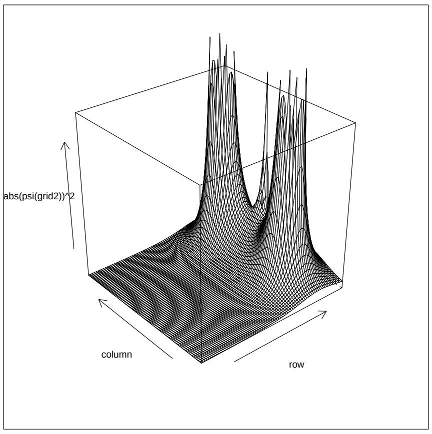


Figure 2: Plot of $z \mapsto |\psi(z)|^2$ on the square $[-1.6, 1.6] \times [-1.6i, 1.6i]$.

```
psi <- function(z) 1/(1 - 0.8*z)
f <- function(x) abs(psi(exp(2i*pi*x)))^2

plotPGram <- function(data, smooth) {
  k <- kernel("daniell", if (smooth) floor(sqrt(length(data))) else 0)
  title <- sprintf("Raw periodogram for %d samples", length(data))
  if (smooth)
    title <- sprintf("Smoothed periodogram for %d samples", length(data))

  p <- spec.pgram(data, k, taper=0, log="no", ylim=c(0,40), main=title)

  grid <- (0:50) / 100
  lines(grid, f(grid), lty=2)

  df <- p$df
  U <- df / qchisq(0.025, df)
  L <- df / qchisq(0.975, df)

  len <- length(p$spec)
  idx <- round(len/5) # Spectral density at 0.1
  # Return a confidence interval
  c(p$spec[idx] * L, p$spec[idx] * U)
}

q4 <- function(n) {
  x <- arima.sim(model=list(ar=0.8), n)
  plotPGram(x, F)
```

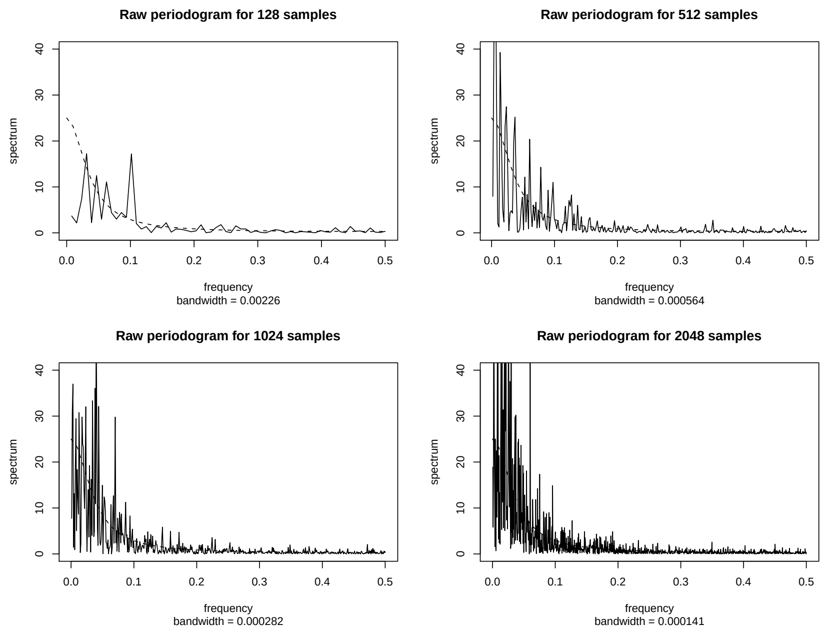


Figure 3: Unsmoothed periodograms at different sample sizes.

```
}

q4(128)
q4(512)
q4(1024)
q4(2048)
```

(b) Our confidence intervals for the four simulations were $[4.66, 678.47]$, $[0.87, 126.45]$, $[0.76, 110.90]$ and $[1.30, 189.51]$. Clearly, we are not very confident in the unsmoothed periodogram, even for large samples sizes. This is consistent with the asymptotic theory, which says that each point of the periodogram has a non-zero asymptotic variance.

5. See Figure 3 for a plot of the periodograms. The code that generated them was (in addition to the code in the previous question):

```
q5 <- function(n) {
  x <- arima.sim(model=list(ar=0.8), n)
  plotPGram(x, T)
}

q5(128)
```

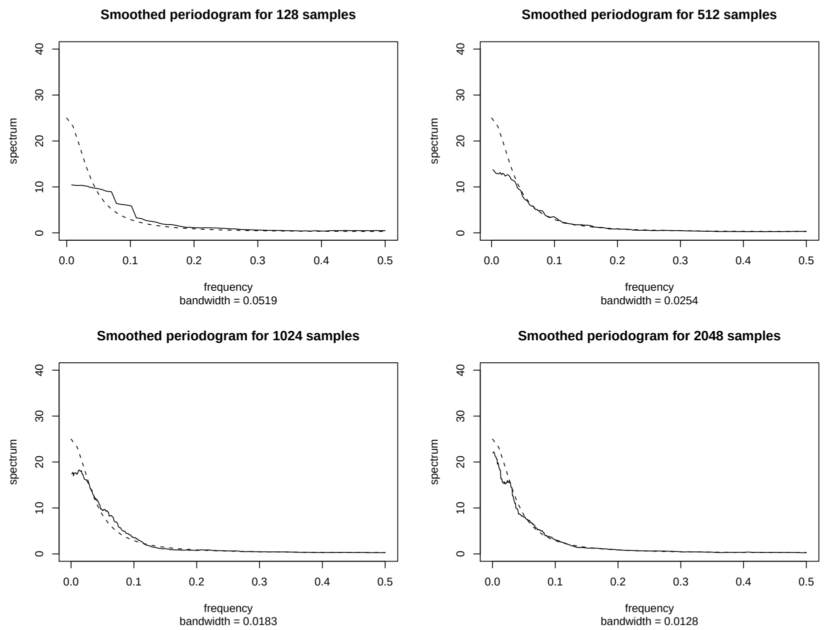


Figure 4: Smoothed periodograms at different sample sizes.

```
q5(512)
q5(1024)
q5(2048)
```