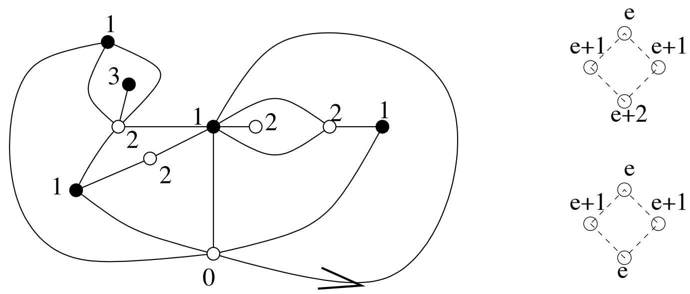

FIGURE 3. Labelling by distance from the root vertex and the two possible configurations of labels (top: a simple face; bottom: a confluent face).

The origin of the root is called the *root vertex*. Two rooted planar maps are considered identical if there exists an homeomorphism *of the plane* that sends one map onto the other (roots included).

The difference between planar graphs and planar maps is that the cyclic order of edges around vertices matters in maps, as illustrated by Figure 2. Observe that planar maps can be equivalently defined on the sphere. In particular Euler's characteristic formula applies and provides a relation between the numbers n of edges, f of faces and v of vertices of any planar map: $f + v = n + 2$.

The *degree* of a face or of a vertex of a map is its number of incidence of edges. A planar map is a *quadrangulation* if all faces have degree four. All (planar) quadrangulations are *bipartite*: their vertices can be colored in black or white so that the root is white and any edge joins two vertices with different colors. In particular a quadrangulation contains no loop but may contain multiple edges. See Figures 1 and 3 for examples of quadrangulations.

Let $\mathcal{Q}_n$ denote the set of rooted quadrangulations with n faces. A quadrangulation with n faces has $2n$ edges (because of the degree constraint) and $n+2$ vertices (applying Euler's formula). The number of rooted quadrangulations with n faces was obtained by W.T. Tutte [34]:

$$(1) \qquad |\mathcal{Q}_n| \;=\; \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}.$$

Various alternative proofs of this result have been obtained (see *e.g.* [10, 13, 5, 31]). Our treatment will indirectly provide another proof, related to [13, 31].

2.2. **Random planar lattices.** Let L_n be a random variable with uniform distribution on $\mathcal{Q}_n$. Formally, L_n is the $\mathcal{Q}_n$-valued random variable such that for all $Q \in \mathcal{Q}_n$

$$\Pr(L_n = Q) \;=\; \frac{1}{|\mathcal{Q}_n|} \;=\; \frac{1}{\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}}.$$

The random variable L_n is our *random planar lattice*. To explain this terminology, taken from physics, observe that locally the usual planar square lattice is a planar map whose faces and vertices all have degree four. Our random planar lattice corresponds to a relaxation of the constraint on vertices.