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A Problem concerning 2-Dimensional Random Posets : notes by Aldous/Brightwell.

A partial order on $[n]$ induces a partial order on any subset $A \subset [n]$. Given two partial orders on $[n]$, say $\preceq^1$ and $\preceq^2$, we can defined their "subset of maximum coincidence" as

$$\text{coin}(\preceq^1, \preceq^2) := \max\{\#A : \preceq^1 \text{ and } \preceq^2 \text{ are identical on } A\}.$$

Given a probability measure μ_n on the space of all partial orders on $[n]$, one can define a random variable $X_n \in \{1, 2, \ldots, n\}$ by

$$X_n = \text{coin}(\preceq^1, \preceq^2),$$

$$\text{where } \preceq^1 \text{ and } \preceq^2 \text{ are chosen uniformly from } \mu_n.$$

We seek to study X_n in the case where μ_n is the random 2-dimensional partial order $\preceq$ on $[n]$. This random partial order has two equivalent descriptions [**?**, **?**, **?**].
(i) Take two independent uniform random permutations π', π'' on $[n]$. Declare $i \preceq j$ iff $\pi'_i \leq \pi'_j$ and $\pi''_i \leq \pi''_j$.
(ii) Take n points (U_i, V_i) distributed uniformly independently in the unit square $[0, 1]^2$. Define $i \preceq j$ iff $U_i \leq U_j$ and $V_i \leq V_j$.

Conjecture 1 *There exists a constant c_0 such that as $n \to \infty$*

$$EX_n \sim c_0 n^{1/3} \text{ and } n^{-1/3} X_n \overset{p}{\to} c_0.$$

We will outline proofs of the bounds

$$\limsup_n n^{-1/3} EX_n \leq c^+ := 3.85 \quad (1)$$

$$\liminf_n n^{-1/3} EX_n \geq c_- := 0.728. \quad (2)$$

It will be clear that the numerical bounds could be improved. Intriguingly, different approaches to bounds lead to different subproblems; we give only suboptimal solutions to the subproblems. Our methods do not seem sharp enough to get the (presumed) correct limit c_0.

0.1 Motivation

If instead of partial orders we consider two uniform random *total* orders on $[n]$, then the analog of X_n has the same distribution as

$L_n :=$ length of longest increasing subsequence of a random permutation of $[n]$,

for which it is known [**?**, **?**] that $EL_n \sim 2n^{1/2}$, and whose more refined properties are of current interest as part of a circle of ideas surrounding the topic of largest eigenvalues of random matrices [**?**, **?**, **?**]. We do not know whether our X_n has any analogous connection to other mathematical structures. Our methods conceptually parallel bounds obtained in the early history ([**?**]; [**?**] section 6.7) of the study of L_n, before it was discovered that the limit constant was 2. Hammersley's ingenious representation of L_n in terms of a two-dimensional Poisson processes enables the subadditive ergodic theorem to be used to prove existence of the limit $\lim_n n^{-1/2}EL_n$, but we do not know any such argument to prove existence of a limit c_0 in Conjecture 1, nor do we have any conjecture for the numerical value of c_0.

0.2 Lower bound 1

Subproblem 2 *Let B_m be the random $\{0,1\}$-valued $m \times m$ matrix whose entries are independent and equal 1 with probability a/m. Let Y_m be the maximal number of 1's in a $\{0,1\}$-valued matrix $A \leq B_m$ such each row and column of A has at most one 1. Show that*

$$m^{-1}EY_m \to \alpha(a)$$

for some explicit constant $\alpha(a)$.

We will prove instead the bound

$$\liminf_m m^{-1}EY_m \geq \alpha_-(a) := a^{-1}(1-e^{-a})^2. \tag{3}$$

Define A by: A has a 1-entry at (i,j) iff B_m has a 1-entry at (i,j) and B_m has no 1-entries at any (i',j) with $i' < i$ or any (i,j') with $j' < j$.

Clearly

$$P(A_{ij} = 1) = \tfrac{a}{m}(1-\tfrac{a}{m})^{i+j-2}$$

and it easily follows that

$$\lim_m m^{-1}E(\text{ number of 1-entries of } A)$$

$$= a \int_0^1 \int_0^1 \exp(-a(x+y))\ dxdy = \alpha_-(a).$$

To obtain the lower bound, fix b, take $m = m(n) \sim bn^{1/3}$ and divide the unit square into m^2 subsquares of side $1/m$. Think of each random partial order as being obtained from n random points in the unit square. The chance that the i'th point falls into the same specified subsquare in both processes $= m^{-4}$. So the event that a given subsquare contains, for *some* i, the i'th point in each process has chance $\sim nm^{-4} \sim b^{-3}m^{-1}$. The indicators of these events form a $m \times m$ matrix, in the format of Subproblem 2 with $a = b^{-3}$ (the slight dependence here has no effect). Clearly

$$X_n \geq \text{ the } Y_m \text{ defined in Subproblem 2}$$

and so in terms of the putative limit $\alpha(a)$ therein,

$$\liminf_n n^{-1/3} EX_n \geq b \liminf_m m^{-1} EY_m = b\alpha(b^{-3}).$$

In terms of the lower bound α_- at (2) we now see that (2) holds for

$$c_- := \sup_b b\alpha_-(b^{-3})$$

which works out numerically as ≈ 0.407. We get a better bound in the next section.

0.3 Lower bound 2

Think of each random partial order as being obtained from n random points in the unit square. Write (u_i^1, v_i^1) and (u_i^2, v_i^2) for the points associated with element i. As in the first lower bound, we find a "coincident" subset $A \subset [n]$ by considering some i's whose two points (in the two processes) are close together; but now we don't discretize.

For real u^1, u^2 write $[u^1, u^2]$ for the interval $[\min(u^1, u^2), \max(u^1, u^2)]$. Say i and j *interfere* if the intervals $[u_i^1, u_i^2]$ and $[u_j^1, u_j^2]$ intersect, or the intervals $[v_i^1, v_i^2]$ and $[v_j^1, v_j^2]$ intersect. If A is a subset such that no i and j in A interfere, then A is a coincident subset.

Write $u_i = |u_i^1 - u_i^2|$ and $v_i = |v_i^1 - v_i^2|$. For small u, v, the conditional distribution of $(u_i^1, v_i^1, u_i^2, v_i^2)$ given $(u_i, v_i) = (u, v)$ is (very close to) the distribution of $(U, V, U \pm u, V \pm v)$ where U and V are independent $U(0,1)$ and the choices of $\pm$ are independent uniform. It is then easy to see

$$P(i \text{ and } j \text{ interfere} | u_i, v_i, u_j, v_j) \approx u_i + v_i + u_j + v_j$$

provided the right side is small. We now define a coincident subset A by first ordering $[n]$ so that $u_i + v_i$ is increasing; then include i in A iff no previously-included j interferes with i.

Within this scheme, write $f_n(t)$ for the chance that an element with $u + v = t$ in not interfered by any previous element. Since the density of points (u, v) equals $4ndudv$, we get an approximation (for small t)

$$1 - f_n(t) = \int\int_{x+y<t} 4ndxdy \ f_n(x + y) \ (x + y + t). \qquad (4)$$

Here $4n \ dxdy \ f_n(x+y)$ gives the density of previously-included points (x, y), so the integral gives an upper bound on the probability that (u, v) is interfered with. So a more precise interpretation of (4) is that we can arrange to include elements in A so that (u, v) is included with probability approximately $f_n(u + v)$.

Setting $x + y = s$, equation (4) becomes

$$1 - f_n(t) = n \int_0^t 4s(s + t)f_n(s) \ ds.$$

Continue until the time t_n^* when $f_n(t) = 0$; then size of A is about

$$n \int_0^{t_n^*} 4sf_n(s) \ ds.$$

Rescale to set $f_n(t) = f(tn^{1/3})$. Then equation above becomes

$$1 - f(t) = \int_0^t 4s(s + t)f(s) \ ds.$$

This can be rephrased as the ODE

$$f'' + 8t^2 f' + 20tf = 0; \quad f(0) = 1, f'(0) = 0.$$

For the coincidence sets A_n constructed this way we have $n^{-1/3}EA_n \to c_-$ for

$$c_- := \int_0^{t^*} 4tf(t) \ dt$$
$$t^* := \inf\{t : f(t) = 0\}.$$

Numerically, $c_- = 0.728$.

0.4 Lower bound 3

Subproblem 3 *In two independent random 2-dimensional partial orders on* $[n]$*, what is the maximum size* Z_n *of a subset* $A \subset [n]$ *such that each partial order restricted to* A *is the trivial partial order (i.e.* A *is an anti-chain in each order)?*

Obviously this Z_n is a lower bound for X_n.

We can find a lower bound for Z_n in a way parallel to that used in the previous section. Think of each random partial order as being obtained from n random points in the unit square. Consider the two points associated with element i, and write x_i and y_i for the absolute distances from those points to the reverse diagonal $u + v = 1$, measured orthogonally. Note the density of (x_i, y_i) is approximately $8dxdy$ for small x, y. For two elements $i, j \in [n]$ a brief calculation shows

$$P(i \text{ and } j \text{ not anti-chain in one order}) \approx \sqrt{2}(\max(x_i, x_j) + \max(y_i, y_j)).$$

when the right side is small. Now order $[n]$ so that $x_i + y_i$ is increasing. Construct a subset A_n which is an anti-chain in each order by the greedy algorithm: inductively, include i if possible.

The mean size EA_n can be analyzed as in the previous section, starting by writing $f_n(x, y)$ for the chance that a point with to-diagonal distances (x, y) is accepted. We end with

$$\lim_n EA_n = c_- := \int\int f(x', y')\ dx'dy'$$

where $f(x, y)$ is the solution of

$$1 - f(x, y) = \int\int_{x'+y'<x+y} f(x', y') 8dx'dy'\ \sqrt{2}(\max(x, x') + \max(y, y')).$$

I haven't tried to evaluate this numerically.

0.5 Upper bound 1

Subproblem 4 *Write* $p(m)$ *for the probability that two independent uniform random 2-dimensional partial orders are identical. Show*

$$p(m) = (\beta + o(1))^m/(m!)^2 \text{ as } m \to \infty$$

for some explicit constant β*.*

It is plausible this holds with $\beta = 1$; we give an upper bound $\beta \leq 3$ below.

To derive an upper bound, for two independent random 2-dimensional partial orders on $[n]$ consider the quantity

$$E(\text{ number of size-}m\text{ subsets where the two partial orders coincide}).$$

This equals $\binom{n}{m}p(m)$; on the other hand it is at least

$$E\binom{X_n}{m} \geq \binom{EX_n}{m}.$$

This inequality implies

$$\max_m \frac{\binom{EX_n}{m}}{\binom{n}{m}p(m)} \leq 1.$$

Since $\binom{n}{m} \leq n^m/m!$, assuming the result of Subproblem 4 for some β gives

$$\max_m \frac{x!}{(x-m)!} \frac{(m!)^2}{n^m(\beta + o(1))^m} \leq 1; \; x = (EX_n)!. \tag{5}$$

Stirling's formula gives

$$n^{-1/3}\log(an^{1/3})! = a(\tfrac{1}{3}\log n + \log a - 1) + o(1).$$

Taking $m \sim bn^{1/3}$ and $x = EX_n \sim cn^{1/3}$ for some $0 < b < c < \infty$, and applying $n^{-1/3}\log(\cdot)$ to each term of (5), a short calculation leads to

$$\sup_{0<b<c} 2b\log b - (c-b)\log(c-b) - 3b - b\log\beta + c\log c \leq 0. \tag{6}$$

This argument, appropriately rephrased, shows that if β is an upper bound in Subproblem 4 then an upper bound c^+ in (1) is given by the smallest c for which (6) holds. Numerically,

$$\text{if } \beta = 3 \text{ then } c^+ = 3.85; \quad \text{if } \beta = 1 \text{ then } c^+ = 2.67.$$

0.6 Upper bound in Subproblem 4.

Here we prove $\beta \leq 3$; precisely

Proposition 5 $p(m) \leq (m+1)3^m/(m!)^2$.

Let $G(n)$ be the number of 4-tuples of linear orders on $[n]$ such that, for each pair i, j from $[n]$, i is below j in exactly two of the four orders (say the 4-tuple is *balanced*). It is easy to check that $G(n)$ is equal to the number of 4-tuples (L_1, L_2, L_3, L_4) such that $L_1 \cap L_2 = L_3 \cap L_4$ (since this is equivalent to (L_1, L_2, L_3^*, L_4^*) being balanced, where L^* denotes the reversed order). So $p(n) = G(n)/n!^4$.

Also let $F(n)$ be the maximum, over all 4-tuples $(L_1, \dots, L_4)$ of linear orders on $[n]$, of the number of 4-tuples $(I_1, \dots, I_4)$ of subsets of $[n]$ such that each I_j is an initial segment of L_j, and every element of $[n]$ occurs in exactly two of the I_j (say $(I_1, \dots, I_4)$ is *equitable*.

Claim 1. $G(n) \leq G(n-1)F(n-1)$.

Claim 2. $F(n) \leq 6\binom{n+2}{2}$.

It follows from these Claims that $G(n) \leq 3^n(n+1)!n!$, so $p(n) \leq (n+1)3^n(n!)^{-2}$.

Proof of Claim 1. For every balanced 4-tuple $(L_1, \dots, L_4)$ of linear orders on $[n]$, we can obtain a balanced 4-tuple of linear orders on $[n-1]$ by deleting the occurrence of n from each linear order. Conversely, given a balanced 4-tuple $(M_1, \dots, M_4)$ of linear orders on $[n-1]$, inserting n into a slot above the initial segment I_j of M_j (for each j) gives a balanced 4-tuple iff every element of $[n-1]$ occurs in exactly 2 of the I_j – i.e., $(I_1, \dots, I_4)$ is equitable. By definition there are at most $F(n-1)$ equitable 4-tuples, and the claim follows.

Proof of Claim 2. By induction on n. True for $n = 1$, since $F(1) = 6 \leq 18$.

Given any fixed 4-tuple $(L_1, \dots, L_4)$ of linear orders on $[n]$, and any equitable 4-tuple $(I_1, \dots, I_4)$, removing n from each linear order (and from the two I_j's in which it appears) leaves a (fixed) 4-tuple $(M_1, \dots, M_4)$ of linear orders on $[n-1]$, and an equitable 4-tuple $(J_1, \dots, J_4)$. There are at most $F(n-1)$ possible equitable 4-tuples $(J_1, \dots, J_4)$. How can a 4-tuple $(J_1, \dots, J_4)$ arise from 2-different equitable 4-tuples $(I_1, \dots, I_4)$? This is possible only if, for two of the linear orders, wlog L_1 and L_2, n is the element immediately above J_1 in L_1 and immediately above J_2 in L_2, so that we could have either $I_1 = J_1 \cup \{n\}$ and $I_2 = J_2$ or vice versa. But this is possible for only at most $n+1$ pairs (I_3, I_4), since $|I_1| + \ldots + |I_4| = 2n$ for all equitable 4-tuples. We now see that the number of equitable 4-tuples $(I_1, \dots, I_4)$ is at most $F(n-1) + 6(n+1) \leq 6\binom{n+1}{2} + 6(n+1) = 6\binom{n+2}{2}$ – 4-tuples $(J_1, \dots, J_4)$ arising from more than two equitable 4-tuples give an overcount here.

Comments. Claim 2 isn't sharp: the right answer must be about $2(n+$

$1)^2$, with the worst case being (L, L, L^*, L^*). It might be possible to reduce this to about n^2, under some suitable additional assumption.