



A species of voter model driven by immigration

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ABSTRACT

A geometric process is proposed in which a random political constituency map emerges through an influx of settlers to a region. At first there is a spatially dispersed native population with established political opinions. The settlers, in sequence, choose random locations in the region and each adopts the political opinion of its closest neighbour, whether native or fellow settler. Various qualitative properties of the resulting constituencies are derived for a planar case of the model. A linear case is also discussed.

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1. Introduction and model

Traditional voter models are part of the interacting particle system tradition and are surveyed in the two books by Liggett (1985, 1999). In these models there is a static configuration of individuals, each of which has a current political allegiance. From time to time an individual changes the colour of its political spots by reference to the opinions of its neighbours.

By contrast, in the model we propose here, the politics of individuals are immutable and the political map emerges through a process of settler immigration. There is an initial, spatially distributed native population with entrenched and varied political views. Settlers arrive one by one, each choosing an independent point in space and each adopting the political colour of its nearest neighbour, whether native or fellow settler. The final political map is then formed by tidying the resulting data.

Let us formalize the general model. The political battleground is a metric space (E, ρ) with Borel σ -field \mathcal{B} . Natives have decided political allegiances, either Black or White, and occupy non-empty disjoint Borel subsets B_0 and W_0 of E . Settlers arrive in sequence, choosing points $(\xi_n)_{n \in \mathbb{N}}$ independently in E according to a non-atomic probability measure μ on E , assuming that such exists. We suppose that $\mu(B_0) = \mu(W_0) = 0$ and that the underlying probability space is complete.

Let B_n and W_n denote the sets of Black and White voters just after the arrival of the n th settler. These are defined inductively as follows:

$$\begin{aligned} \text{if } \rho(\xi_n, B_{n-1}) < \rho(\xi_n, W_{n-1}) \quad \text{set } B_n &= B_{n-1} \cup \{\xi_n\} \quad \text{and} \quad W_n = W_{n-1}; \\ \text{if } \rho(\xi_n, B_{n-1}) > \rho(\xi_n, W_{n-1}) \quad \text{set } B_n &= B_{n-1} \quad \text{and} \quad W_n = W_{n-1} \cup \{\xi_n\}. \end{aligned}$$

Here $\rho(\xi, B) = \inf_{x \in B} \rho(\xi, x)$ denotes the distance between $\xi \in E$ and $B \in \mathcal{B}$. Ties, for which $\rho(\xi_n, B_{n-1}) = \rho(\xi_n, W_{n-1})$, are to be broken independently at random, though these will not figure in the cases we study.

Let

$$B_\infty = \bigcup_{n=0}^{\infty} B_n, \quad W_\infty = \bigcup_{n=0}^{\infty} W_n.$$

From these raw data we colour the political map. Let B'_∞ be the set of accumulation points of B_∞ ; likewise, W'_∞ . The final Grey, Black and White constituencies, which may not exhaust E , are then defined as follows:

$$\mathbf{G} = B'_\infty \cap W'_\infty, \quad \mathbf{B} = B'_\infty \setminus \mathbf{G}, \quad \mathbf{W} = W'_\infty \setminus \mathbf{G}.$$

The aim is to understand the nature of these constituencies.

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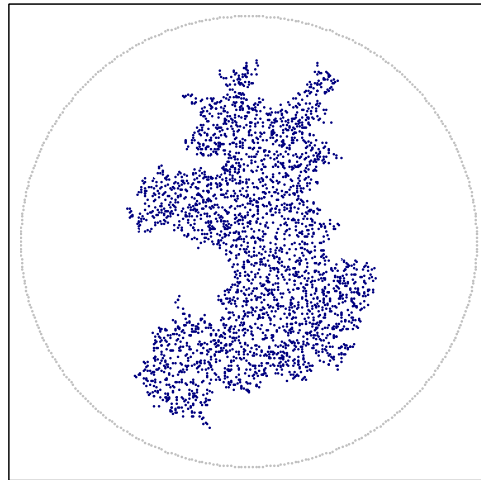


Fig. 1.

It is hard to believe that the above voter–settler model is new, for it certainly poses some intriguing questions; nevertheless, we have found no appearance in the probability literature. Perhaps the closest relatives are Voronoi processes (Borovkov and Odell, 2007), since at each stage of the settler sequence E is divided into coloured Voronoi cells whose geometry governs future developments. However, such processes seem generally monochrome. A two-population model with some remote similarity is discussed in Deijfen and Häggstrom (2004).

We suggest applications in biology, sociology, cognitive mapping and linguistics: (a) a botanical species populates a region, each new plant being infected by one of two mutually excluding contagions; (b) from time to time members of a social network purchase a certain item, each choosing the brand of a close associate; (c) a child acquires concepts, classifying them ‘attractive’ or ‘repulsive’, say, by reference to cognate concepts; (d) settlements develop in a geographical area and inhabitants adopt a linguistic trait, e.g., a vowel sound, from their nearest neighbours. As a concrete but fanciful example (which we do not study here), let E be the sphere S^2 , with the usual metric, and let μ be normalized Lebesgue measure on E . Let B_0 and W_0 consist of north and south poles, respectively. Settlers regard themselves as ‘northerners’ or ‘southerners’ according to the designation of their respective nearest neighbours. The Black, White and Grey constituencies are then political ‘northern’ and ‘southern’ hemispheres together with the ‘equator’.

Rather than attempting a general theory in this introductory note, we study two specific cases. In the first and principal one the settlers choose sites uniformly in the unit disc with White circumference and Black centre. Section 2 is devoted to the proof of the main tool in this investigation, and Section 3 gives some basic topological features of the resulting constituencies. The second case, which is the subject of Section 4, is a variant of the classical ‘sticking breaking’ model (Lloyd, 1989) which, incidentally, has enjoyed a resurgence of attention recently owing to its applications in Bayesian statistics. Here E is the unit interval with native Black and White end-points. In the final map a Grey point constituency separates Black and White sub-interval constituencies, and the question is: what is the distribution of this point?

2. The planar case

In this section and the next we work in \mathbb{R}^2 with the usual metric. The closed disc with centre $x \in \mathbb{R}^2$ and radius $r > 0$ will be denoted by $S(x, r)$ and its circumference by $C(x, r)$. We also write $S(r) = S(0, r)$ and $C(r) = C(0, r)$ with 0 being the origin. Specifically, we study what seems to be a reasonable prototype – the voter–settler model for a Black-centred disc with a White circumference:

$$E = S(1), \quad \mu = \text{normalized Lebesgue measure on } E, \quad B_0 = \{0\}, \quad W_0 = C(1). \quad (*)$$

From now on we work exclusively in the topology of E .

For the purposes of acclimatization, Fig. 1 shows a single realization of the set B_n with $n = 10\,000$; the White points are invisible. The process is restless, so there is no guarantee that the ostensibly Black region will not be infiltrated significantly by White points later on. Nevertheless one would expect the final constituency map for this realization to be dominated by a single large region of each colour, separated by a highly complex border region. Such casual observations are not easy to translate into rigorous propositions.

The mainstay of our work is the result below, which allays the fear that all points will end up Grey. The remainder of this section is devoted to the proof.

Theorem 1. Consider the voter–settler model with data (*). Then

$$\mathbb{P}(0 \in \mathbf{B}) = 1.$$

Throughout the argument, C will denote a generic positive constant. Introduce discs $S_k = S(2^{-k})$, $k = 0, 1, \dots$, and write $\mathcal{E}_n = \{\xi_1, \dots, \xi_n\}$ for the settler locations at stage n .

Lemma 1. *Let $m \geq 0$ be an integer. Then, almost surely, for some (random) $k, n \geq 0$, $|\mathcal{E}_n \cap S_{k+1}| = |\mathcal{E}_n \cap S_k| = m$.*

Proof. If the first m settlers lie in S_1 , stop. Otherwise, restrict attention to a disc S_j that excludes these m points and try again, and so on. These Bernoulli trials have success probability $(1/4)^m$, and almost surely one will give the required condition. \square

Because of Lemma 1, on inflating S_k , it is more than enough to prove the following.

Lemma 2. *In (*) replace $B_0 = \{0\}$ with $B_0 = \{0, \eta_1, \eta_2, \dots\}$, where the η_i are points of an independent Poisson process on $E = S(1)$ with intensity $\lambda > 0$. Then*

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}(S(1/2) \subset \mathbf{B}) = 1. \tag{1}$$

A further lemma is needed before the proof of Lemma 2. To prepare for this let $T \sim \text{Poisson}(\pi\lambda)$ be independent of the ξ_i and η_i sequences. At stage T the η_i together with ξ_1, \dots, ξ_T constitute a Poisson process of intensity 2λ on E . We shall show that when λ is large, White points are unlikely to penetrate very far towards 0.

Lemma 3. *Assume the conditions of Lemma 2. Let $r \in (0, 1)$ and let*

$$A = \bigcup_{s \in S(r)} \{\rho(s, B_T) > \rho(s, W_T)\},$$

where T is defined above. Then

$$\mathbb{P}(A) \leq q(\lambda, r) := 1 \wedge C\lambda^{2/3}(1-r)^{-2} \exp\{-C(1-r)^2\lambda^{1/3}\}. \tag{2}$$

Proof. The argument is combinatorial. Define the annulus $\mathcal{A} = S(1) \setminus S(r)$, and write

$$u = C(1), \quad v = S(r), \quad \mathcal{E} = \mathcal{E}_T, \quad H = \{\eta_1, \eta_2, \dots\}.$$

Say that a subset of \mathcal{A} is η free if it contains no points of H . Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the random directed graph with vertex set $\mathcal{V} = \{u, v, \{\xi_1\}, \dots, \{\xi_T\}\}$ and edge set \mathcal{E} , where $(x, y) \in \mathcal{E}$ if and only if

$$x \neq y \text{ and } \exists s \in y \text{ s.t. } \rho(s, x) < \rho(s, H),$$

i.e., whenever x and y are distinct and some point of y is closer to some point of x than to all η_i . As usual, the in-degree of $y \in \mathcal{V}$ is

$$\text{indeg}(y) = \#\{x \in \mathcal{V} : (x, y) \in \mathcal{E}\}.$$

An n -path is a directed path in \mathcal{G} from u to v through n points of \mathcal{E} ; in particular, if $(u, v) \in \mathcal{E}$ this edge constitutes a 0-path. Write

$$\Delta_v = \text{indeg}(v), \quad \Delta = \max_{\xi \in \mathcal{E}} \text{indeg}(\{\xi\}),$$

so that the number of n -paths is no larger than $\Delta_v \Delta^n$.

Next we prove some probability bounds. First, since $\Delta_v \leq \#\mathcal{E}$ and $\#\mathcal{E} \sim \text{Poisson}(\pi\lambda)$ we have

$$\mathbb{P}(\Delta_v > 7\lambda) \leq \mathbb{P}(\#\mathcal{E} > 7\lambda) \leq e^{-7\lambda} \mathbb{E}e^{\#\mathcal{E}} = e^{-7\lambda} \cdot e^{\pi\lambda(e-1)} \leq e^{-C\lambda}. \tag{3}$$

Secondly, observe that, conditional on the set $X = \mathcal{E} \cup H$, the points in X belong to \mathcal{E} or H according to independent coin flips. Hence, for the in-degree of a point x in \mathcal{V} to exceed M , both x and the M points nearest to x (if such exist) must lie in \mathcal{E} . Thus,

$$\begin{aligned} \mathbb{P}(\Delta > M) &= \mathbb{E}\mathbb{P}(\Delta > M | X) \\ &= \mathbb{E} \left(\mathbb{P} \left(\bigcup_{x \in X} ((x \in \mathcal{E}) \cap (\text{indeg}(\{x\}) > M)) \mid X \right) \right) \\ &\leq \mathbb{E}(\#X \cdot (1/2) \cdot 2^{-M}) = 2^{-M} \lambda. \end{aligned} \tag{4}$$

Thirdly, for $N \in \mathbb{N}$, writing $r' = 1 - r$,

$$\mathbb{P}(\exists \text{ an } n\text{-path for some } 0 \leq n < N) \leq C(N/r')^2 e^{-C\lambda(r'/N)^2}. \tag{5}$$

For, if there is an n -path, $0 \leq n < N$, then there must be at least one η free set in \mathcal{A} of the form $\mathcal{A} \cap S(x, r'/N)$, $x \in \mathcal{A}$. But it is possible to choose a collection of fewer than $C(N/r')^2$ discs of radius $r'/4N$ with centres in \mathcal{A} so that at least one is contained in each of the aforementioned sets. The probability that any one of these discs is η free is less than $e^{-C\lambda(r'/N)^2}$, and (5) follows.

We are now in a position to complete the argument. Motivated by (3)–(5), define

$$B = B_{N,M,\lambda} = (\Delta_v \leq 7\lambda) \cap (\Delta \leq M) \cap (\nexists \text{ an } n\text{-path for any } 0 \leq n < N).$$

Call an n -path *live* if the settlers thereon arrive in the order prescribed by the path. Thus, each n -path is live with probability $1/n!$, independently of the sets \mathcal{E} and H . Let

$$L_n = \text{the number of live } n\text{-paths, } n = 0, 1, \dots, \quad L = \sum_{n=0}^{\infty} L_n.$$

Note that the event A occurs precisely when there is a live n -path for some $n \geq 0$. Thus,

$$\mathbb{P}(A) = \mathbb{P}(L \geq 1) \leq \mathbb{P}(L \geq 1 \cap B) + \mathbb{P}(B^c). \tag{6}$$

For the first term on the right of (6) we have

$$\mathbb{P}(L \geq 1 \cap B) \leq \mathbb{E}(L \mathbf{1}_B) = \sum_{n=N}^{\infty} \mathbb{E}(L_n \mathbf{1}_B) \leq \sum_{n=N}^{\infty} \frac{7\lambda M^n}{n!} \leq \frac{7\lambda M^N}{N!} e^M \leq C\lambda(eM/N)^N e^M,$$

where conditioning on \mathcal{G} is needed for the second inequality and Stirling’s approximation for the fourth. For the second term on the right of (6), by virtue of (3)–(5),

$$\begin{aligned} \mathbb{P}(B^c) &\leq \mathbb{P}(\Delta_v > 7\lambda) + \mathbb{P}(\Delta > M) + \mathbb{P}(\exists \text{ an } n\text{-path for some } 0 \leq n < N) \\ &\leq e^{-C\lambda} + \lambda e^{-CM} + C(N/r')^2 e^{-C\lambda(r'/N)^2}. \end{aligned}$$

Finally, in order to use these estimates efficiently in (6), choose $N = \lfloor \lambda^{1/3} \rfloor$, and $M = \lfloor N/10 \rfloor$. This yields

$$\mathbb{P}(A) \leq C\lambda^{2/3} r'^{-2} e^{-Cr'^2 \lambda^{1/3}}. \quad \square$$

Proof of Lemma 2. Define new discs by $S_k = S(r_k)$, $k \in \mathbb{N}_0$, with $r_0 = 1$, $r_k \downarrow 1/2$. For each k let T_k be a random variable such that the η_i together with ξ_1, \dots, ξ_{T_k} form a Poisson process of intensity $2^k \lambda$ on $E = S(1)$. Write

$$A_k = \bigcup_{s \in S_k} \{\rho(s, B_{T_k}) > \rho(s, W_{T_k})\}, \quad C_k = \bigcap_{i=1}^k A_i^c, \quad A^* = \bigcap_{i=1}^{\infty} A_i^c,$$

and observe that $A^* \subset \{S(1/2) \subset \mathbf{B}\}$.

We know already from Lemma 3 that

$$\mathbb{P}(A_1) \leq q(\lambda, r_1). \tag{7}$$

Since the event C_k depends only on the ξ_i and η_i in the annulus $S_0 - S_k$, it follows that, conditional on C_k , the ξ_i and η_i in S_k at time T_k still constitute a Poisson process of intensity $2^k \lambda$ and are, moreover, coloured Black. Paying proper regard to scaling we can therefore re-use the estimate from Lemma 3 to deduce that

$$\mathbb{P}(A_{k+1} | C_k) \leq q(2^k \lambda r_k^2, r_{k+1}/r_k), \quad k \in \mathbb{N}. \tag{8}$$

We have, therefore,

$$\mathbb{P}(A^*) = 1 - \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = 1 - \left(\mathbb{P}(A_1) + \sum_{k=1}^{\infty} \mathbb{P}(A_{k+1} | C_k)\right) \geq 1 - \sum_{k=0}^{\infty} q(2^k \lambda r_k^2, r_{k+1}/r_k),$$

and as long as the r_k sequence does not converge too rapidly (e.g., $r_k = (k + 1)/2k$) the last summation vanishes as $\lambda \rightarrow \infty$. This proves (1) and, with it, Theorem 1. \square

By an obvious coupling argument, the conclusion of Theorem 1 remains valid when the native locations are changed to $B_0 = \{0\} \cup B_0^+$ and $W_0 = W_0^+$, where for some $0 < \epsilon < 1$, $(B_0^+ \cup W_0^+) \cap S(\epsilon) = \emptyset$. We have been unable to determine whether $\mathbb{P}(C(1) \subset \mathbf{W}) = 1$, but the following is clearly true.

Corollary 1. Under the conditions of Theorem 1, if $w \in C(1)$, then $\mathbb{P}(w \in \mathbf{W}) = 1$.

3. The planar constituencies

We continue to study instance (*) of the voter–settler model described in Section 2, the aim now being to use Theorem 1 to prove some qualitative properties of the constituencies.

Theorem 2. For the voter–settler model with data (*), each of the following holds almost surely in the topology of E .

1. The sets \mathbf{B} , \mathbf{W} and \mathbf{G} partition E .
2. \mathbf{B} is an open set containing B_∞ ; \mathbf{W} is an open set containing $W_\infty \setminus W_0$.
3. $\mathbf{G} = \partial\mathbf{B}$ is non-empty, closed, dense in itself and nowhere dense.
4. $\mu(\mathbf{G}) = 0$.

Proof. We omit a.s. qualifications; complements are with respect to E .

1. This is an immediate consequence of $B_\infty \cup W_\infty$ being almost surely dense in E .
2. Let b_n be the n th point of B_∞ to occur. Consider a disc in E with centre b_n , small enough to exclude all previous points, both settler and native. Apply Theorem 1 to settlers in this disc to conclude that $b_n \in \mathbf{B}$. Hence, $B_\infty \subset \mathbf{B}$.
Further, if $x \in \mathbf{B}$ there is an open disc S with centre x that contains no point of W_∞ . Because $B_\infty \cup W_\infty$ is dense in S , $S \subset \mathbf{B}$, and therefore \mathbf{B} is open.
The properties of \mathbf{W} follow similarly.
3. First of all, let Q be a fixed, countable subset of W_0 that is also dense therein; write $Z = Q \cup (W_\infty \setminus W_0)$. As in 2., $Z \subset \mathbf{W}$.
Now let $x \in E$ and let V be any open neighbourhood of x . If $x \in \mathbf{G}$ then V contains a point in $B_\infty \subset \mathbf{B}$ and a point in $Z \subset \mathbf{B}^c$; hence $x \in \partial\mathbf{B}$. Conversely, if $x \in \partial\mathbf{B}$ then V contains a point of \mathbf{B} and a point of \mathbf{B}^c . It follows that V contains a point of B_∞ and a point of $Z \subset W_\infty$, and so $x \in \mathbf{G}$. Thus, $\mathbf{G} = \partial\mathbf{B}$.
Since \mathbf{G} is the boundary of an open set it is both closed and nowhere dense. Also, by Theorem 1 and Corollary 1 we know that $(0, 0) \in \mathbf{B}$ and $(1, 0) \in \mathbf{W}$. Hence $(a, 0) \in \mathbf{G}$ when $a = \inf\{x > 0 : (x, 0) \in \mathbf{W}\}$, i.e., \mathbf{G} is not empty.
Finally, any open disc with centre $g \in \mathbf{G}$ contains a point $b \in \mathbf{B}$ and a point $w \in Z \subset \mathbf{W}$. In this disc an arc that contains b and w but not g must include a further point in \mathbf{G} , and therefore \mathbf{G} is dense in itself.
4. To verify that $\mu(\mathbf{G}) = 0$ it suffices by Fubini’s theorem to show that $\mathbb{P}(x \in \mathbf{G}) = 0$ for all $x \in E \setminus W_0$. Accordingly, pick $\lambda > 0$ and $x \in E \setminus W_0$, and let $S_k = S(x, 2^{-k})$, $k \in \mathbb{N}$, and $T \sim \text{Poisson}(\lambda)$, independently. By Lemma 1, for some (random) $k \geq 1$, every one of the first T settlers to arrive in S_k will lie in S_{k+1} ; hence they will each have the same colour. We are then in a position to conclude from Lemma 2 that $\mathbb{P}(x \in \mathbf{G}) = 0$ (the fact that $0 \in B_0$ in this lemma has no bearing on its conclusion). \square

Next we confirm the complexity of the constituencies.

Theorem 3. For the voter–settler model with data (*), almost surely, \mathbf{B} , \mathbf{W} and \mathbf{G} are disconnected and each has an infinity of connected components.

To this end, fix $n \in \mathbb{N}$ and let $b \in B_n$, $w \in W_n$. Write

$$r = \rho(b, w), \quad I = S(b, r) \cap S(w, r), \quad U = S(b, r) \cap S(w, r).$$

We say that b and w are cocooned if the sets $S(b, r) \cap W_n$ and $S(w, r) \cap B_n$ are empty: for instance, the closest pair of opposing points at any stage is cocooned. Observe that the only coloured points in I are b and w , and that $S(b, r) \subset E$ since $C(1) \subset W_n$. The key to Theorem 3 is the abundance of cocooned pairs.

Lemma 4. Let $N \in \mathbb{N}$. Almost surely, at some stage in the settler process there are at least N cocooned pairs whose respective U are disjoint.

Proof. Let A be one of N congruent closed sectors of $S(1)$. For each $n \in \mathbb{N}$ let b_n, w_n be the closest pair of points for which $b_n \in A \cap B_n$ and $w_n \in W_n$ (almost surely such a pair is unique). Then $\rho(b_n, w_n) \rightarrow 0$. Since A is compact we can extract a convergent subsequence $b_{n'}$. Almost surely, the limit of this subsequence is not 0, by Theorem 1, and does not lie on one of the bounding radii of A since the phase of each $b_n \neq 0$ is uniformly distributed over the range appropriate to A . It follows that, almost surely, for all large enough n' the points $b_{n'}$ and $w_{n'}$ are cocooned and their U lies in A . Repeating the argument, restricting attention to the subsequence n' for each sector in turn, yields the required cocooned pairs. \square

Now suppose that at a certain stage points b and w are cocooned. For $\epsilon > 0$ consider the following potential developments regarding subsequent settlers in U (see Fig. 2):

- (a) the first settler b' in U lies in $S(w, r) \cap S(b, \frac{1}{2}r) \subset I$ and is thus coloured Black;
- (b) the next settlers x_1, x_2, \dots in U are coloured White, forming the successive vertices of a polygon in I that circumscribes b' , with the distance between each pair of adjacent vertices being less than $r\epsilon$;
- (c) once the x_i have arrived we intervene in the process and colour Black the circle $C(b, r)$, including the point w , and then let the process resume;
- (d) for each i , $S(x_i, r\epsilon) \subset \mathbf{W}$.

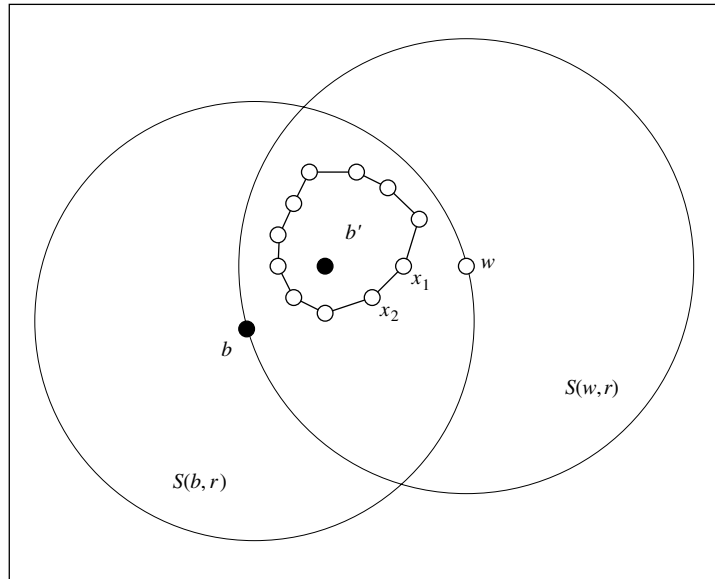


Fig. 2.

The purpose of the intervention in (c) is revealed in the argument below.

Proof of Theorem 3. Let $F(\epsilon)$ denote the event that (a), (b) and (d) occur, subject to (c), and let $F_0(\epsilon) \supset F(\epsilon)$ denote the same event without the intervention. For all sufficiently small ϵ we have

$$\mathbb{P}(F_0(\epsilon)) \geq \mathbb{P}(F(\epsilon)) \geq \delta(\epsilon) > 0, \tag{9}$$

where $\delta(\epsilon)$ does not depend on r or the geometry of $U \cap E$. Fix such a value of ϵ .

If $F(\epsilon)$ occurs we say that b' is *encircled*. In this case, by virtue of Theorem 1, the polygon in (b), which is a subset of \mathbf{W} , circumscribes at least one connected component of each of \mathbf{B} and \mathbf{G} , almost surely. Furthermore, under the intervention, if there are N cocooned pairs in E with disjoint U then the respective events $F(\epsilon)$ are *independent*, and so the number of encircled Black points is at least Binomial($N, \delta(\epsilon)$). In view of (9) the same holds without the intervention. A similar argument applies to encircling White points. Since, by Lemma 4, we may choose N arbitrarily large, Theorem 3 follows immediately. \square

In closing this section we suggest three problems.

Problem 1. Find the distribution of $\mu(\mathbf{B})$.

Problem 2. Find the distributions of the inner and outer radii of \mathbf{B} :

$$R_* = \rho(0, \mathbf{W}), \quad R^* = 1 - \rho(C(1), \mathbf{B}).$$

Problem 3. Find the (presumably degenerate) distribution of the Hausdorff dimension of \mathbf{G} .

4. The linear case

In this section we work in \mathbb{R} with the usual metric, taking

$$E = [0, 1], \quad \mu = \text{Lebesgue measure on } E, \quad W_0 = \{0\}, \quad B_0 = \{1\}.$$

Thus the ξ_n are i.i.d. Uniform(0, 1). It is clear that, almost surely,

$$\mathbf{G} = \{Y\}, \quad \mathbf{W} = [0, Y] \quad \mathbf{B} = (Y, 1],$$

where Y is a random variable taking values in $(0, 1)$. We identify the distribution of Y in our final result.

Theorem 4. $Y \sim \text{Beta}(2, 2)$ with density $6y(1 - y)$, $0 < y < 1$.

Proof. If $\xi_1 > 1/2$ then $W_1 = \{0\}, B_1 = \{\xi_1\}$ and Y is determined by the analogous voter–settler process on $[0, \xi_1]$; similarly for $\xi_1 < 1/2$. Accordingly, Y satisfies the distributional equation

$$Y \stackrel{d}{=} \begin{cases} \xi Y & \text{if } \xi > 1/2, \\ \xi + (1 - \xi)Y & \text{if } \xi < 1/2, \end{cases} \quad \xi \sim \text{Uniform}(0, 1), \quad Y \perp \xi. \tag{10}$$

Eq. (10) can be rewritten as the stochastic linear equation

$$Y \stackrel{d}{=} AY + B, \quad Y \perp (A, B), \tag{11}$$

where

$$A = \max\{\xi, 1 - \xi\} \sim \text{Uniform}(1/2, 1), \quad B = \xi \mathbf{1}(\xi < 1/2).$$

Despite a developed theory, explicit solutions of such equations are rare, especially when, as here, A and B are not independent. Nevertheless it follows from Theorem 1.5 and 3.2 of [Vervaat \(1979\)](#) (the latter being due to Grincevičius) that the distribution of Y is determined uniquely by (11) and is continuous; moreover the required distribution is clearly symmetric about $1/2$. The plausible $Y \sim \text{Beta}(2, 2)$ satisfies (10), and thus [Theorem 4](#) is proved. \square

This rings a little hollow: can the solution of (11) be found more systematically? One approach is to use the Laplace transform $\psi(s) := \mathbb{E}e^{-sY}, s > 0$. From (11) we have

$$\begin{aligned} \psi(s) &= \mathbb{E}e^{-s(A Y + B)} = \mathbb{E}\mathbb{E}(e^{-s(A Y + B)} | \xi) \\ &= \mathbb{E}(e^{-sB} \psi(sA)) \\ &= \int_0^{1/2} e^{-sx} \psi(s(1-x)) dx + \int_{1/2}^1 \psi(sx) dx \\ &= \int_0^{1/2} (e^{-sx} + 1) \psi(s(1-x)) dx; \end{aligned}$$

and putting $u = sx$ yields the following integral equation for ψ :

$$s\psi(s) = \int_0^{s/2} (e^{-u} + 1) \psi(s-u) du, \quad s > 0.$$

This is a homogeneous Volterra equation of the second kind with a discontinuous kernel. The ‘half-convolution’ form looks inviting, but it is not mentioned in [Polyanin and Monzhirov \(1998\)](#), for example, and seems to lie outside the analytical mainstream. [Theorem 4](#) provides a probabilistic solution.

A different approach is to start from scratch and condition on ξ_1 . This yields the following functional equation for F , the distribution function of Y :

$$F(x) = \begin{cases} \int_0^x F\left(\frac{x-u}{1-u}\right) du + \int_{1/2}^1 F\left(\frac{x}{u}\right) du, & 0 < x \leq 1/2, \\ 1 - F(1-x), & 1/2 < x < 1. \end{cases} \tag{12}$$

With due care at $x = 1/2$ one can deduce from (12) that $f = F'$ exists, is continuous on $(0, 1)$ and satisfies

$$f'(x) = \frac{1}{(1-x)} f(x) + \frac{1}{x} (f(2x) - f(x)), \quad 0 < x < 1/2, \tag{13}$$

with $f(x) = f(1-x), 1/2 < x < 1$. Consequently, except perhaps at inverse powers of 2, f has derivatives of all orders. At $x = 1/3$, where $f(x) = f(2x)$, the derivatives can be obtained recursively using (13): in fact, the third and higher derivatives are all zero, and the formal Taylor expansion about $x = 1/3$ recommends f as the $\text{Beta}(2, 2)$ density.

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