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SPECIAL INVITED PAPER

ORIENTED PERCOLATION IN TWO DIMENSIONS[1]

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This paper is a self-contained survey of most of the results known about oriented percolation. The table of contents below should give an idea of the topics which will be covered. A more detailed account can be found in the first section.

1. A few words of explanation. Oriented percolation is one of the simplest systems which exhibits a "phase transition," so it has been studied by a number of people using a variety of methods (rigorous and not). At this point in the development of the theory there are a number of results but there are still quite a few open problems, so we have written this article with the hope of acquainting people with what is known and more important with what is not known. With the last purpose in mind, this paper is aimed at two audiences: (1) people who are experts on the subject and (2) people who are not but would like to learn something about it or the broader topic: interacting particle systems. We have separate advice for these two groups on how to read the paper.

(1). Experts will find the main new results in Sections 4–5, 7, 10, 11, 13 and 14. As for the rest of the paper, the material in Sections 3, 6, and 8 is classical, being from the period surveyed in Griffeath (1981). Most of the results in Sections 9–13 are from Durrett and Griffeath (1983) but there have been a number of improvements. The most notable of these are: (i) In four places ((7.1), (7.6), (10.3), and (11.1)) large deviations results replace upper bounds; (ii) We show that (12.4) for $p > p_c$, $P(n < |C_0| < \infty) \ge c \exp(-\Gamma n^{1/2})$, in contrast to (7.6) for

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$p < p_c$, $P(n < |C_0| < \infty) \le C \exp(\gamma n)$; (iii) We show that the edge speed α is continuous for $\lambda > \lambda_c$ (see (10.1)); and (iv) We have made some minor progress toward a central limit theorem for $|\xi_n^0|$, see (13.4).

(2). For the uninitiated, these notes are intended to be an easy to read self-contained introduction to oriented percolation. As we explain in Section 2, this is a subject which we consider to be the study of one particular discrete time interacting particle system, so you can think of this as also an introduction to the larger subject as well. In this regard I think oriented percolation is a good example for two reasons: (i) it can be constructed by flipping coins instead of talking about generators or Poisson processes and (ii) there is a strict upper bound of 1 on the propagation speed so arguments for oriented percolation are simpler than those for its continuous time analogue—the basic contact process.

So much for the how and why. In Section 3 we give a characterization of p_c which is the key to the developments which follow. After you finish Section 3 the following groups of sections may be read more or less independently: 4–5, 6, 7, 8–9, 10–14, subject only the restriction that 9 is a prerequisite for the last group. I must confess that I have an unnatural fascination with computing lower bounds and refining the results in the subcritical case so if the focus gets too microscopic you should feel free to skip ahead.

Finally two words about NOTATION: The important formulas in each section are numbered (1), (2), $\cdots$. As above when we are in another section and want to refer to (6) of Section 7 we will call it (7.6). There are two important announcements about the constants C and γ in Section 7.

2. Description of the model. From each $z \in Z^2$ there is an oriented arc to $z + (0, 1)$ and to $z + (1, 0)$ (see Figure 1a.) Each arc, also called a bond, is independently open with probability p and closed with probability $1 - p$. We think of open bonds as permitting us to go along the bond in the direction of the orientation and with this in mind we make the following definitions:

$x \to y$ (y can be reached from x) if there is an open path from x to y: that is, there is a sequence $x_0 = x, x_1, \cdots, x_m = y$ of points in Z^2 such that for each $k \le m$ the arc from x_{k-1} to x_k is open

C_0(the cluster containing $0 \equiv (0, 0)) = \{x : 0 \to x\}$

C_0 is the set of all points we can reach from 0.

In percolation the event of interest is

$$\Omega = \{|C_0| = \infty\}$$
$$= \{\text{there is an infinite open path starting at } 0\},$$

(here and below $|A|$ is the number of points in A). The reason for our interest in Ω_∞ is that its probability is 0 if p is small and positive if p is close to 1, i.e., as the value of p increases the system undergoes a "phase transition" from having only finite clusters to having an infinite connected set.

When this model was introduced by Broadbent and Hammersley in (1957) it was used to explain a phenomenon which is well known and useful to people who

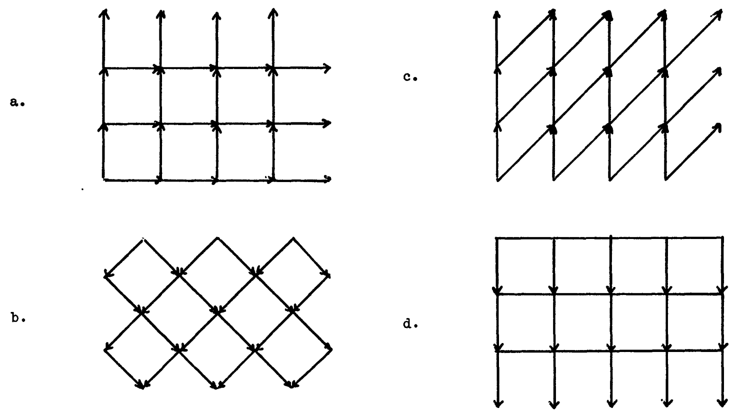

FIG. 1.

drill for oil: A given liquid will penetrate some rocks but not others. In the model open bonds correspond to airspaces which permit the passage of fluid, the probability p is related to how porous the material is, and the appearance of an infinite cluster indicates that the liquid can freely penetrate the solid. In this application the orientation given in Figure 1a is of course ridiculous and an unoriented model is usually used. We would like to point out, however, that if we rotate the lattice $-135°$ (see Figure 1b) or better yet consider the partially oriented model given in Figure 1d (this is the "stiff" model), we then get a model which can incorporate the effect of gravity on the movement of the fluid.

We can model another type of system if we change the lattice by sending $(x, y) \to (x, x + y)$ (see Figure 1c) and interpret $\xi_n^0 = \{(x, n) \in C_0\}$ as the set of occupied sites at time n. If we generalize the percolation scheme so that bonds up ($\uparrow$) are open with probability p and bonds at 45° ($\nearrow$) are open with probability p' then the dynamics of the process ξ_n^0 may be described as follows: At each time n each particle $x \in \xi_n^0$ independently (a) dies with probability $1 - p$ and, (b) gives birth to a new particle at $x + 1$ with probability p', subject to the restriction (c) there is never more than one particle per site.

If we allow an unlimited number of particles per site we get what is called a branching random walk. It is known (see Athreya and Ney, 1972) that in this case $E\,|\,\xi_n^0\,| = (p + p')^n$ and if $p + p' > 1$ then the number of particles at 0 is $\sim C(p + p')^n/n^{1/2}$. The first result is the familiar Malthusian law of exponential population growth, but the second conclusion shows that the density of particles becomes so great that the assumption that the particles live and die independently is not valid.

Oriented percolation is a small first step in taking into account that an enviroment can support only a limited density of individuals. This model takes the very drastic approach of limiting the density to one indivdiual per site but as we shall see below this very special case is already very complicated, so we will stick to to this model below and leave it to the reader to try to generalize the results to more realistic interactions.

The interpretations mentioned above are just two of many. Oriented percolation, by virtue of its simplicity, has appeared as a model of processes that occur in chemistry (Schlögl, 1972, Grassberger and de la Torre, 1979), and in connection with Reggeon field theory which models the creation, propogation and destruction of a cascade of elementary particles (see Moshe, 1978, and Cardy and Sugar, 1980). The three dimensional model (= 2 space + 1 time) has been used to model the evoluation of galaxies with the hope of explaining the appearance of spiral arms (see papers of Gerola, Seiden, and Schulman) and to describe "hopping conduction in an amorphous semiconductor" (van Lien and Shklovskii, 1981).

The interpretations listed above are just mentioned to convince you that oriented percolation is a widely used model. Apart from potential applications there is one final reason for interest in oriented percolation—it is in a different "universality class" than regular percolation. What this means is that the nature of the infinite cluster and the behavior of the system for p near p_c is different in the two models. We will say something about these differences below. The similarities are apparent and I think the reader will find it interesting to compare the results here with those for ordinary percolation in Kesten's *Percolation Theory for Mathematicians.*

The key to the theory in Kesten's book is the idea of a sponge crossing, which is analogous to the right edge discussed in Section 3 and in each case the characterization of p_c in terms of this object. For some purposes, like getting bounds on $P(|C_0| \geq n)$ below p_c, the work is much easier in the oriented case. When we turn to study the model on the square lattice at p_c, however, the unoriented case is easier—it has two axes of symmetry, so all the sponge crossing probabilities are the same. In the oriented case when p is near p_c percolation occurs in a narrow cone, so the sponge crossings do not satisfy the hypotheses of Kesten's metatheorem, and we cannot use it to conclude $P(\Omega_\infty) = 0$ at p_c.

3. A characterization of p_c. In this section we will give a way of characterizing p_c which is the key to the developments which follow. First we rotate our picture 45°. Let $\mathscr{L} = \{(m, n) \in Z^2 : m + n \text{ is even}, n \geq 0\}$ and draw an oriented arc from each $(m, n) \in \mathscr{L}$ to $(m + 1, n + 1)$ and to $(m - 1, n + 1)$ (The picture is just Figure 1b upside down). Let $\xi_n^0 = \{x : (x, n) \in \mathscr{L} \text{ and } 0 \to (x, n)\}$. ξ_n^0 is a random subset of $\{-n, \cdots, n\}$.

$$\text{Let } r_n = \sup \xi_n^0. \qquad (\sup \varnothing = -\infty)$$

$$\text{Let } \ell_n = \inf \xi_n^0. \qquad (\inf \varnothing = +\infty)$$

r_n and ℓ_n stand for the right edge and left edge of ξ_n^0.

$$\text{Let } \bar{\xi}_n = \{x : (x, n) \in \mathscr{L} \text{ and there is a } y \leq 0$$

$$\text{such that } (y, 0) \to (x, n)\}.$$

(Note: in the last definition I should say $(y, 0) \in \mathscr{L}$ but here and in what follows I will omit this and it will be understood that all points referred to are in $\mathscr{L}$).

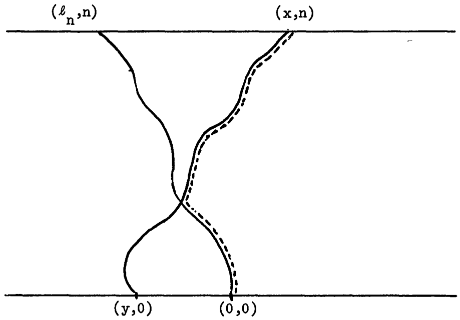

FIG. 2

Let $\bar{r}_n = \sup \bar{\xi}_n$ (there are an infinite number of starting points so $\bar{\xi}_n \neq \varnothing$). It is clear that $\bar{r}_n \geq r_n$. Our interest in $\bar{r}_n$ comes from the following facts

(1) $$\xi_n^0 = \bar{\xi}_n \cap [\ell_n, \infty)$$

(2) $$\text{on } \{\xi_n^0 \neq \varnothing\}, \quad r_n = \bar{r}_n.$$

PROOF. The second result is a corollary of the first so we will only prove (1). Clearly $\xi_n^0 \subset \bar{\xi}_n$ and $\xi_n^0 \subset [\ell_n, \infty)$ so we only have to show $\xi_n^0 \supset \bar{\xi}_n \cap [\ell_n, \infty)$ and to do this we can suppose $\ell_n < \infty$. In this case it is clear from a picture (see Figure 2) that if there is a path to $(x, n) \geq \ell_n$ from some $(y, 0)$ with $y < 0$ then there is also a path from 0 to (x, n), so $x \in \xi_n^0$. (Note: Here it is important that $d = 2$ and that when two paths cross they must have at least one vertex in common.)

REMARK. The reader who does not believe that the argument given above is painful enough to be a rigorous proof can verify (as I did in Durrett, 1980) that (1) holds at time 0 and every possible transition preserves this equality. In (10) below we will state some other coupling results without proof. They can all be proved by drawing a picture like (1) or using induction.

From (2) we see that to study the asymptotic behavior of r_n it suffices to consider $\bar{r}_n$. The latter process has two advantages (a) $\bar{r}_n > -\infty$ for all n and (b) it can be embedded in a nice two-parameter process:

$$\text{let } \bar{r}_{m,n} = \sup\{x - \bar{r}_m : (x, n) \in \mathscr{L} \text{ and there is}$$
$$\text{a } y \leq \bar{r}_m \text{ so that } (y, m) \to (x, n)\}.$$

Figure 3 should help explain the definition. In the outcome we have drawn the lines represent open bonds and all the other bonds are closed.

From the definition it is clear that

(3) $$\{\bar{r}_{m+1,n+1} : 0 \leq m < n\} =_d \{\bar{r}_{m,n} : 0 \leq m < n\}$$

and from the picture it is clear that

(4) $$\bar{r}_m + \bar{r}_{m,n} \geq \bar{r}_n.$$

To prove (4) we observe that $\bar{r}_m + \bar{r}_{m,n}$ is the rightmost point on the line $y = n$ which can be reached from some (x, m) with $x \leq \bar{r}_m$ whereas $\bar{r}_n$ is the rightmost point on the line $y = n$ which can be reached from some (x, m) which can be reached from some $(y, 0)$ with $y \leq 0$ and (hence) has $x \leq \bar{r}_m$.

It is unfortunate that the argument above cannot be extended to conclude that if $\ell < m < n$

(4′) $$\bar{r}_{\ell,m} + \bar{r}_{m,n} \geq \bar{r}_{\ell,n}.$$

(Figure 4 gives a counterexample) because if (4′) held we could use Kingman's (1973) ergodic theorem to conclude that as $n \to \infty$

(5) $$\bar{r}_{0,n}/n \to \alpha \quad \text{a.s.},$$

where $\alpha \in [-\infty, 1]$ is a constant given by

(6) $$\alpha = \inf_{n \geq 1} E(\bar{r}_n/n).$$

All is not lost however. The process $\bar{r}_{m,n}$ is subadditive enough so that we can

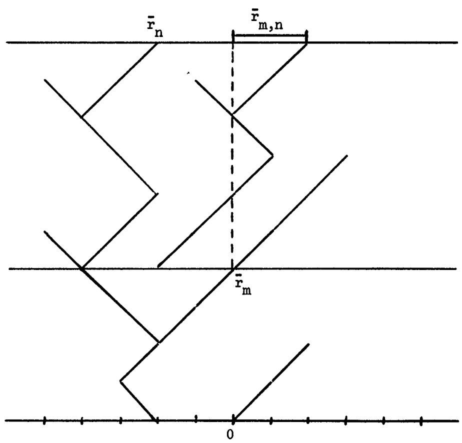

FIG. 3.

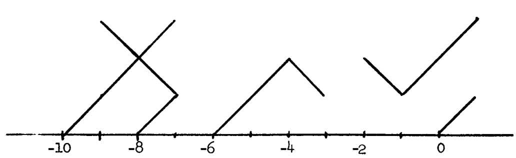

FIG. 4. $\bar{r}_1 = 1$; $\bar{r}_{1,2} = -1$; $\bar{r}_{1,3} = 0$; $\bar{r}_2 = -5$; $\bar{r}_{2,3} = -3$.

imitate the proof of Kingman's theorem. Since the details of this proof have already appeared once in this journal (see Durrett, 1980), and soon will appear again in a slightly more general form (see Liggett, 1985) we will not go into them here.

As a corollary of (6) and (2) we get

$$\text{(7)} \qquad \text{On } \Omega_\infty \quad r_n/n \to \alpha \quad \text{a.s.} \quad \ell_n/n \to -\alpha \quad \text{a.s.}$$

Since $\ell_n \le r_n$ on Ω_∞, it follows that if $P(\Omega_\infty) > 0$ then $\alpha \ge -\alpha$ i.e. $\alpha \ge 0$ or turning the last statement around

$$\text{(8)} \qquad \text{If } \alpha < 0 \quad \text{then} \quad P(\Omega_\infty) = 0.$$

In the other direction it is easy to show

$$\text{(9)} \qquad \text{If} \quad \alpha > 0 \quad \text{then} \quad P(\Omega_\infty) > 0.$$

PROOF. The keys to the proof are (i) the trivial observation that if $\alpha > 0$ then $\bar{r}_n \to \infty$ a.s. and hence there is an even integer $M < \infty$ so that

$$P(\bar{r}_n > -M \text{ for all } n) \ge .51$$

and (ii) a generalization of the "coupling result" given in (1). The rest of the proof is just introducing notation:

$$\text{if} \quad A \subset (-\infty, \infty) \quad \text{we let}$$

$$\xi_n^A = \{x\text{: there is a } y \in A \text{ with } (y, 0) \to (x, n)\}$$

(recall our convention that all points referred to are in $\mathscr{L}$)

$$r_n^A = \sup \xi_n^A; \qquad \ell_n^A = \inf \xi_n^A;$$

$$\tau^A = \inf\{n : \xi_n^A = \varnothing\}.$$

Repeating the proof of (1) shows that for any every $M \ge 0$

$$\text{(10a)} \qquad \xi_n^{[-M,M]} = \xi_n^{(-\infty,M]} \cap [\ell_n^{[-M,M]}, \infty)$$

$$\text{(10b)} \qquad \xi_n^{[-M,M]} = \xi_n^{[-M,\infty)} \cap (-\infty, r_n^{[-M,M]}]$$

$$\text{(10c)} \qquad \xi_n^{[-M,M]} = \xi_n^{(-\infty,\infty)} \cap [\ell_n^{[-M,M]}, r_n^{[-M,M]}]$$

so on $\xi_n^{[-M,M]} \ne \varnothing$

$$\text{(10d)} \qquad r_n^{[-M,M]} = r_n^{(-\infty,M]}, \qquad \ell_n^{[-M,M]} = \ell_n^{[-M,\infty)}$$

and it follows that

$$\text{(10e)} \qquad \tau^{[-M,M]} = \inf\{n : r_n^{[-M,M]} < \ell_n^{[-M,M]}\} = \inf\{n : r_n^{(-\infty,M]} < \ell_n^{[-M,\infty)}\}.$$

From (10e) we see immediately that

$$\{\tau^{[-M,M]} = \infty\} \supset \{\ell_n^{[-M,\infty)} \le 0 \le r_n^{(-\infty,M]} \text{ for all } n\}.$$

Since it is obvious that

$$P(r_m^{(-\infty,M]} > 0 \text{ for all } m) = P(r_m^{(-\infty,0]} > -M \text{ for all } m) \geq .51$$

it follows that $P(\xi_n^{[-M,M]} = \varnothing$ for all $n) \geq .02$. Since $P(\xi_{M/2}^0 \supset 2\mathbb{Z} \cap [-M, M]) > 0$ it follows that $P(\Omega_\infty) > 0$.

REMARK. The argument above is simple but it does not lead to a reasonable or even explicit lower bound on $P(\Omega_\infty)$. In Section 5 we will show that $P(\Omega_\infty) \geq \alpha(p)^2$ and hence applying (12), $P(\Omega_\infty) \geq 4(p - p_c)^2$.

Since $\alpha(p)$ is a nondecreasing function of p, we have almost shown

(11) $$p_c = \inf\{p:\alpha(p) > 0\}.$$

(8) and (9) imply

$$\sup\{p:\alpha(p) < 0\} \leq p_c \leq \inf\{p:\alpha(p) > 0\}$$

so to complete the proof of (11) we have only to rule out the possibility that $\{p:\alpha(p) = 0\}$ is an interval of positive length. This is not as easy as it sounds. To prove (11) we will show

(12) If $\alpha(p') > -\infty$ then $\alpha(p) - \alpha(p') \geq 2(p - p')$.

The first step in showing (12) is to show

(13) If $A \supset B$ are infinite subsets of $\{-2, -4, \cdots\}$ then

$$E(r_n^{B\cup\{0\}} - r_n^B) \geq E(r_n^{A\cup\{0\}} - r_n^A) \geq 2.$$

PROOF. From the definition of ξ_n^S it is immediate that

$$\xi_n^{C\cup D} = \xi_n^C \cup \xi_n^D$$

hence

$$r_n^{C\cup D} = r_n^C \vee r_n^D \quad (a \vee b = \max\{a, b\})$$

and

$$r_n^{C\cup D} - r_n^C = 0 \vee (r_n^D - r_n^C) = (r_n^D - r_n^C)^+.$$

From the above we see

$$r_n^{B\cup\{0\}} - r_n^B = (r_n^{\{0\}} - r_n^B)^+ \geq (r_n^{\{0\}} - r_n^A)^+ \quad (r_n^A \geq r_n^B)$$
$$= r_n^{A\cup\{0\}} - r_n^A$$

To get the last inequality in (13) observe that if $A = \{-2, -4, \cdots\}$ then by translation invariance

$$E(r_n^{\{0,2,\cdots\}} - r_n^{\{-2,-4,\cdots\}}) = 2.$$

The argument above is due to Tom Liggett. Now that we have established (13) it is fairly routine to finish the proof of (11). The next step is to show

(14) If $p > p'$ and we let $\alpha_n(p) = E\bar{r}_n$ then

$$\alpha_n(p) - \alpha_n(p') \geq 2(1 - (p - p'))^n.$$

PROOF. Construct the systems with parameters p and p' on the same space in the obvious way: assign an independent random variable U_b to each bond b which is uniformly distributed on (0, 1) and call the bond open if $U_b <$ the parameter value and closed otherwise. Let $\bar{r}_n$ and $\bar{r}'_n$ be the location of $\sup \xi_n^{(-\infty,0]}$ in the systems with parameters p and p' respectively. Let $\tau = \inf\{n: \bar{r}_n > \bar{r}'_n\}$ and construct a third system $\bar{\xi}''_n$ in which $\bar{\xi}''_0 = \{0, -2, -4, \cdots\}$, the parameter is p for bonds in $\{(x, y): y < \tau\}$ and p' for bonds in $\{(x, y): y > \tau\}$. If $\bar{r}''_n = \sup \bar{\xi}''_n$ then

$$\bar{r}_\tau = \bar{r}''_\tau \geq \bar{r}'_\tau + 2$$

and

$$\bar{\xi}_\tau = \bar{\xi}''_\tau \supset \bar{\xi}'_\tau \cup \{\bar{r}_\tau\}$$

so applying (13) and the strong Markov property gives

$$E\bar{r}_n - E\bar{r}'_n \geq E\bar{r}''_n - E\bar{r}'_n \geq 2P(\tau \leq n)$$

which implies the desired result since

$$P(\tau \leq n) \geq (1 - (1 - (p - p'))^n).$$

(At each stage there is probability $p - p'$ that $\bar{r}_n \to \bar{r}_n + 1$ while $\bar{r}'_n \to \bar{r}'_{n+1} < \bar{r}'_n + 1$. Since $\bar{r}_n$ can get ahead of $\bar{r}'_n$ in other ways this is only an inequality).

If we divide both sides of the inequality in (14) by n and let $n \to \infty$ we do not get what we want, so we have to resort to a trick. Let $\delta = (p - p')/M$ where M is a large integer

$$\alpha_n(p) - \alpha_n(p') = \sum_{m=1}^{Mn} \alpha_n\left(p' + \frac{m\delta}{n}\right) - \alpha_n\left(p' + \frac{(m-1)\delta}{n}\right)$$
$$\geq Mn \cdot 2\left(1 - \left(1 - \frac{\delta}{n}\right)^n\right).$$

Dividing both sides of the inequality by n and letting $n \to \infty$ we see

$$\alpha(p) - \alpha(p') \geq 2M(1 - e^{-(p-p')/M})$$

(here we use the fact that $\alpha(p) \geq \alpha(p') > -\infty$ so there is convergence in L^1 in (6). Letting $M \to \infty$ in the last inequality gives

$$\alpha(p) - \alpha(p') \geq 2(p - p').$$

REMARKS. As $p \uparrow 1$, $1 - \alpha(p) \sim 2(1 - p)$ so the result above is sharp. (12) and the fact that $\alpha(p)$ is continuous for $p > p_c$ (which we will prove in Section 11) would be immediate if we knew that $\alpha(p)$ was concave for $p > p_c$.

It is clear that $p \to \alpha(p)$ is upper semicontinuous (i.e. a decreasing limit of continuous functions) and hence that $\alpha(p)$ is continuous from the right. The proof of left continuity for $p > p_c$ is more difficult but is an easy consequence of the construction in Section 9.

4. A stationary distribution for the edge process. In this section we will show

(1) If $p \geq p_c$ then there is an initial distribution μ concentrated on the infinite subsets of $\{\cdots -4, -2, 0\}$ which contain 0, so that r_n^μ has stationary increments and we will study some of the properties of μ. The reason for interest in this result will become clear when we apply the result in the next section.

Idea behind the proof. If we let $\tilde{\xi}_n = \bar{\xi}_n - \bar{r}_n = \{x - r_n : x \in \bar{\xi}_n\}$, i.e., what we see if we stand at the right edge of $\bar{\xi}_n$, then $\tilde{\xi}_n$ is a Markov chain on the set of subsets of $\{\cdots, -4, -2, 0\}$ which contain 0. The state space is compact so if the transition mechanism for this process were Feller, we could quote a well-known result to establish the existence of stationary distribution. The transition mechanism is not Feller however (it is not continuous at $\{0\}$), so we apply the technique of proof (make Cesaro averages and take subsequential limits) rather than the result itself. The idea is simple, but as the reader will see below, the details are a little tedious. If you get bored or struck, feel free to skip ahead. The proof is not important for what follows.

PROOF OF (1). Modifying the approach of the last section a little we start by introducing a family of "reset approximations" $\bar{\xi}_m^n$ which start from $\bar{\xi}_0^n = (-\infty, 0]$ and evolve according to the following rules:

(i) if $(m + 1) \notin nZ$ the transition from $\bar{\xi}_m^n$ to $\bar{\xi}_{m+1}^n$ is made as before.
(ii) if $(m + 1) \in nZ$ then we let $\bar{\xi}_{m+1}^n = (-\infty, \bar{r}_{m+1}^n]$ where $\bar{r}_{m+1}^n$ is the rightmost point on level $m + 1$ which can be reached from $\{(x, m) : x \in \bar{\xi}_m^n\}$.

To construct μ we will take a limit of the reset processes. The increments $X_k^n = \bar{r}_k^n - \bar{r}_{k-1}^n$ of these processes are not stationary but they are periodic so if we introduce an independent r.v. U_n with $P(U_n = k) = 1/n$ for $0 \leq k < n$ then the shifted increments $Y_k^n = X_{k+U_n}^n$ are a stationary sequence with

$$EY_1^n = n^{-1} \textstyle\sum_{k=1}^n EX_k^n = n^{-1} E\, \bar{r}_k^n \geq \alpha.$$

Since $X_k^n \leq 1$ we have $E(Y_1^n)^+ \leq 1$. Combining with the fact that $0 \leq \alpha = E(Y_1^n)^+ - E(Y_1^n)^-$ gives $E(Y_1^n)^- \leq 1 - \alpha \leq 1$ and we have $E\,|\,Y_1^n\,| \leq 2$. From the last inequality it follows that if we consider the sequence of processes $\{Y_m^n\; m \geq 1\}$ as a sequence of random elements of $R \times R \times \cdots$ then the sequence is tight (see Billingsley, 1968, page 19) so we can find a sequence $n_j \to \infty$ so that $\{Y_m^{n(j)}, m \geq 1\}$ converges weakly (in $R \times R \times \cdots$) to a limit $\{Y_m, m \geq 1\}$ with $E\,|\,Y_m\,| \leq 2$.

For the purposes of Durrett (1980) the limit Y_k, $k \geq 1$, was good enough. To construct μ we have to take another subsequence

$$\text{let} \quad \tilde{\xi}_m^n = \bar{\xi}_{m+U_n}^n - \bar{r}_{U_n}^n \quad \text{where} \quad \bar{r}_{U_n}^n = \sup \bar{\xi}_{U_n}^n$$

and we have used the obvious notation for translating a set S by a constant c, $S - c = \{x - c : x \in S\}$.

$$\text{Let} \quad \tilde{r}_m^n = \sup \tilde{\xi}_m^n \quad \text{and} \quad \tilde{Y}_m^n = \tilde{r}_m^n - \tilde{r}_{m-1}^n.$$

It follows from the definition of the $\tilde{\xi}_m^n$ that $\{\tilde{Y}_m^n, m \geq 1\} =_d \{Y_m^n, m \geq 1\}$.

Thanks to the translation by $\bar r^n_{U_n}$, $\tilde\xi^n_0$ is a subset of $(-\infty, 0]$ and $0 \in \tilde\xi^n_0$. On $\{U_n = k\}$ the Markov property and the translation invariance of the mechanism implies $\tilde\xi^n_m$ evolves like oriented percolation for $0 \le m < n - k$. For fixed K $P(U_n > n - K) \to 0$ so as $n \to \infty$ the finite dimensional distributions of the $\tilde\xi^n_m$ become indistinguishable from those of $\xi^{\mu_n}_m$ where μ_n is the distribution of $\tilde\xi^0_n$.

The μ_n are probability measures on the compact space $\{0, 1\}^{\{\cdots -4,-2,0\}}$ so the sequence $\mu_{n(j)}$ has a further subsequence $\mu_{n(j)}$ which converges to a limit μ. It is easy to see that $\mu_{n(j)} \Rightarrow \mu$ implies that for each M the distribution of $\{\tilde\xi^{n(j)}_m\ 0 \le m \le M\}$ converges weakly to $\{\xi^\mu_m\ 0 \le m \le M\}$. Combining this observation with results above shows that if we let r^μ_m and $X^\mu_m = r^\mu_m - r^\mu_{m-1}$ $m \ge 1$ then $\{X^\mu_m, m \ge 1\} =_d \{Y_m, m \ge 1\}$ so the increments of r^μ_n are a stationary sequence and we have constructed μ.

To complete the proof of (1) we have to show that μ concentrates on configurations with infinitely many ones. This is easy. The argument above shows $E(r^\mu_1)^- \le 1$ so if $-V_n$ gives the location of the nth point in ξ^μ_0 to the left of 0 (we set $V_0 = 0$) then

$$P(r^\mu_1 \le -M - 1 \mid V_n = M) \ge (1 - p)^{2n}$$

so

$$\frac{1}{M + 1} \ge P(r^\mu_1 \le -M - 1,\ V_n = M) \ge (1 - p)^{2n} P(V_n = M).$$

The first thing we need to prove that r^μ_1 is that it has the mean we expect. To do this we observe $-Y^n_1 \ge -1$ so it follows from Fatou's lemma that

$$E(-Y_1) \le \lim\ \inf_{n\to\infty} E(-Y^n_1)$$

i.e.

$$E(Y_1) \ge \lim\ \sup_{n\to\infty} E(Y^n_1) \ge \alpha.$$

On the other hand $r^\mu_n \le \bar r_n$ so dividing by n and letting $n \to \infty$ gives

$$\lim\sup\ (1/n) r^\mu_n \le \alpha \quad \text{a.s.}$$

and the ergodic theorem implies that as $n \to \infty$

$$(1/n) r^\mu_n \to E(r^\mu_1 \mid \mathscr{I}) \equiv E(Y_1 \mid \mathscr{I})$$

where $\mathscr{I}$ is the shift invariant σ-field (see Theorem 6.21 in Breiman, 1968) so $EY_1 = EE(Y_1 \mid \mathscr{I}) \le \alpha$ and it follows that $EY_1 = \alpha$ and $E(Y_1 \mid \mathscr{I}) = \alpha$ a.s.

An important property of μ is that it is a stationary distribution for the process "viewed from the rightedge" i.e. if we let $r^\mu_m = \sup \xi^\mu_m$ and $\tilde\xi^\mu_m = \xi^\mu_m - r^\mu_m$ then $\tilde\xi^\mu_m$ is a stationary sequence (To prove this we observe that by definition $\tilde\xi^n_m$ has this property and as $j \to \infty$ $\tilde\xi^{n(j)}_m$ converges weakly to $\tilde\xi^\mu_m$).

For studying the behavior of $\bar\xi_n$ it would be nice to know if the stationary distribution μ for the edge process is unique or better yet if $\bar\xi_n - \bar r_n$ converges to μ as $n \to \infty$. (The first result would be useful because it would imply (by a standard argument) that the Cesaro averages $((1/n) \sum^n_{m=1})$ of the distributions of $\bar\xi_m - \bar r_m$ would converge.). The difficulty in proving results about $\bar\xi_m - \bar r_m$ seems to be the fact that the process seen from the right edge is not a monotone function of the initial configuration so we cannot use the monotonicity arguments of Section 8 to conclude the existence of limits.

REMARK. From one point of view all we have done in proving (1) is to decompose the subadditive process $\bar{r}_{m,n}$ into an additive process

$$A_{m,n} = \sum_{m<k\leq n} Y_k$$

and a nonnegative subadditive process $Z_{m,n} = \bar{r}_{m,n} - A_{m,n}$ with

$$(1/n)EZ_{0,n} \to 0.$$

This is just the decomposition Kingman (1973) uses to prove his subadditive ergodic theorem. It would be interesting to know if the decomposition was unique in this case, but this is a harder question than determining whether or not μ is the only stationary measure for the process viewed from the edge.

PROBLEM. (related to the idea behind the proof). Is there a stationary distribution for $\tilde{\xi}_n$ for $p < p_c$? Note that there is one for $p = p_c$!

5. Recurrence properties of $\bar{r}_n$, lower bounds on $P(\Omega_\infty)$. In this section we will discuss the two subjects indicated in the title. The first result we will prove is a version of the Kesten, Spitzer, Whitman theorem which is valid for random walks with stationary increments, i.e. it is possible to remove the original hypothesis that the increments are independent.

(1) Let $X_1, X_2 \cdots$ be a stationary sequence of r.v.'s with values in R^d and define a random walk by $S_n = X_1 + \cdots + X_n$. Let $R_n = |\{S_1, \cdots, S_n\}|$ i.e. the number of sites visited in the first n steps and let $A = \{S_1 \neq 0, S_2 \neq 0, \cdots\}$. Then as $n \to \infty$

$$R_n/n \to E(1_A \mid \mathscr{I}) \quad \text{a.s.},$$

where $\mathscr{I}$ is the σ-field of shift invariant events.

PROOF. The proof follows the one given in Breiman (1968) page 121–122 with some minor modifications to make up for the lack of independence and some simplifications which come from taking a more general viewpoint. There are two parts to the proof

$$\text{I.} \quad \liminf_{n\to\infty} R_n/n \geq E(1_A \mid \mathscr{I})$$

$$\text{II.} \quad \limsup_{n\to\infty} R_n/n \leq E(1_A \mid \mathscr{I}).$$

I. Suppose $X_1, X_2 \cdots$ are constructed on the canonical space $R^d \times R^d \times \cdots$ with $X_n(\omega) = \omega_n$, and the shift operator defined as $(\theta\omega)_n = \omega_{n+1}$, and $\mathscr{I} = \{B: 1_B \circ \theta = 1_B \text{ a.s.}\}$ (see Chapter 6 of Breiman, 1968). Clearly

$$R_n \geq \sum_{m=0}^{n-1} 1_A(\theta^m \omega)$$

(the right-hand side only counts points in $\{S_1, \cdots S_n\}$ which are not in $\{S_{n+1}, S_{n+2} \cdots\}$), so it follows from the ergodic theorem (Theorem 6.21 in Breiman,

1968) that

$$\liminf_{n\to\infty} R_n/n \geq E(1_A \mid \mathscr{I}).$$

II. Let $A_N = \{\omega : \omega_1 \neq 0, \omega_1 + \omega_2 \neq 0, \cdots \omega_1 + \cdots \omega_N \neq 0\}$. Clearly

$$R_n \leq \sum_{m=0}^{n-N} 1_{A_N}(\theta^m \omega) + N.$$

Letting $n \to \infty$ and using the ergodic theorem again gives

$$\limsup_{n\to\infty} R_n/n \leq E(1_{A_N} \mid \mathscr{I}).$$

As $N \to \infty$, $A_N \downarrow A$ so it follows from the dominated convergence theorem for conditional expectations that as $N \to \infty$

$$E(1_{A_N} \mid \mathscr{I}) \downarrow E(1_A \mid \mathscr{I}).$$

(1) has a number of interesting corollaries. The most famous is

(2) If X_1 is integer valued and $E(X_1 \mid \mathscr{I}) = 0$ then $P(A) = 0$.

PROOF. If $E(X_1 \mid \mathscr{I}) = 0$ then the ergodic theorem implies that $S_n/n \to 0$ a.s. and it follows easily that

$$(\sup_{m\leq n} S_m)/n, \ (\inf_{m\leq n} S_m)/n \to 0 \quad \text{a.s.}$$

Since the X_k are integer valued we see that

$$R_n \leq 1 + (\sup_{m\leq n} S_m - \inf_{m\leq n} S_m)$$

so $R_n/n \to 0$ a.s., and it follows that $E(1_A/\mathscr{I}) \equiv 0$ and $P(A) = 0$.

REMARK. (2) is a little known well-known result. It was first proved by Atkinson (1976), and later reproved by Derrienic (1980) and Dekking (1982). I learned about the proof above (due to P. van der Vecht) from H. Berbee.

As a consequence of (1) and (2) we get some useful results about oriented percolation.

(3) If $p \geq p_c$ then $P(r_n^\mu > 0 \text{ for all } n \geq 1) = \alpha(p)$.

PROOF. By monotonicity that it suffices to prove the result when $\alpha(p) > 0$. If we let $X_m = r_m^\mu - r_{m-1}^\mu$ for $m \geq 1$ then $S_n = r_n^\mu$ and $S_n/n \to \alpha(p) > 0$ a.s. so we have

$$(\inf_{m\leq n} S_m)/n \to 0, \quad (\sup_{m\leq n} S_m)/n \to \alpha(p).$$

It follows from the fact that S_n does not increase by more than 1 at any time that

$$[1, \sup_{m\leq n} S_m] \subset R_n \subset [\inf_{m\leq n} S_m, \sup_{m\leq n} S_m]$$

and $A = \{r_n^\mu > 0 \text{ for all } n \geq 1\}$, (here we use the fact $r_n^\mu \to \infty$ a.s.) so (1) implies

$$R_n/n \to \alpha(p) = E(1_A \mid \mathscr{I}) \quad \text{a.s.,}$$

and it follows that $P(A) = EE(1_A \mid \mathscr{I}) = \alpha(p)$.

From (3) it follows immediately that we have

(4) $$P(\bar{r}_n > 0 \text{ for all } n \geq 1) \geq \alpha(p),$$

and by symmetry if $\bar{\ell}_n = \inf \xi_n^{[0,\infty)}$ then

(5) $$P(\bar{\ell}_n < 0 \text{ for all } n \geq 1) \geq \alpha(p).$$

When both of the events in (4) and (5) happen it follows from the basic coupling result that $\xi_n^0 \neq \varnothing$ for all n. To combine these estimates to get a lower bound on $P(\Omega_\infty)$ we use an inequality of Harris (1960).

(6) Let f and g be increasing functions on $\{0, 1\}^n$ (i.e. if $x_i \leq y_i$ for $i = 1, \cdots n$ then $f(x) \leq f(y)$). If $X = (X_1, \cdots X_n)$ is a vector of independent $\{0, 1\}$ valued random varibles then

$$Ef(X)g(X) \geq Ef(X)Eg(X).$$

PROOF. If $n = 1$ and $x, y \in \{0, 1\}$ then

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

(there are two cases to consider: $x = 1, y = 0$ and $x = 0, y = 1$), so if X and Y are independent and have values in $\{0, 1\}$

$$\begin{aligned} 0 &\leq E[(f(X) - f(Y))(g(X) - g(Y))] \\ &= Ef(X)g(X) - Ef(Y)Eg(X) - Ef(X)Eg(Y) + Ef(Y)g(Y). \end{aligned}$$

If X and Y have the same distribution it follows that

$$0 < 2Ef(X)g(X) - 2Ef(X)Eg(X)$$

proving the result when $n = 1$.

To prove the result in general we will take conditional expectation and use induction. The key to the proof is that

$$E(f(X) \mid X_1 = y) = Ef(y, X_2, \cdots X_n)$$

is an increasing function of y and $f(y, \cdot, \cdots, \cdot)$ is a function of $n - 1$ variables so we have

$$\begin{aligned} E(f(X)g(X)) &= EE(f(X)g(X) \mid X_1) \\ &\geq E\{E(f(X) \mid X_1)E(g(X) \mid X_1)\} \\ &\geq E\{E(f(X) \mid X_1)\}E\{E(g(X) \mid X_1)\} \\ &= Ef(X) \cdot Eg(X). \end{aligned}$$

REMARK. This should be called Harris' FKG inequality to distinguish it from Harris' (1977) positive correlations inequality. To see why we call it Harris' FKG inequality look at Fortuin, Kastelyn, and Ginibre (1971) or at Batty and Bollman (1980) for more recent results and references.

From (4), (5), and (6) it follows immediately that we have

$$P(\Omega_\infty) \geq \alpha(p)^2 \geq 4(p - p_c)^2 \tag{7}$$

the second inequality coming from (3.12). We think that

$$P(\Omega_\infty) \geq C(p - p_c) \tag{8}$$

but we have not been able to prove this.

In deriving the lower bounds for $P(\Omega_\infty)$ we used the fact that (3) implies

$$P(\bar{r}_n > 0 \text{ for all } n \geq 1) \geq \alpha(p).$$

We can get another interesting result by observing that it also implies

$$P(r_n^0 > 0 \text{ for all } n \geq 1) \leq \alpha(p) \tag{9}$$

$$P(\ell_n^0 < 0 \text{ for all } n \geq 1) \leq \alpha(p). \tag{10}$$

In Section 9 we will show that $\alpha(p_c) = 0$ so if $P(\Omega_\infty) > 0$ then on Ω_∞ r_n^0 and ℓ_n^0 each will return to 0 infinitely often but $r_n^0 - \ell_n^0 \to \infty$. This is absurd and one can almost prove it is a contradiction. Let μ^* be distribution of $-\xi_0^\mu$. r_n^μ and $\ell_n^{\mu^*}$ are random walks with stationary increments which have $E(r_n^\mu/\mathscr{I}) = 0$ and $E(\ell_n^{\mu^*}/\mathscr{I}) = 0$ so we have $E(r_n^\mu - \ell_n^{\mu^*}/\mathscr{I}) = 0$. Unfortunately $r_n^\mu - \ell_n^{\mu^*}$ does *not* have stationary increments so we cannot apply (2) to conclude that $r_n^0 - \ell_n^0 \leq r_n^\mu - \ell_n^{\mu^*} = 0$ infinitely often.

With a little more work one can show that if $P(\Omega_\infty) > 0$ then on Ω_∞, $r_n^0 \to \infty$ in probability (this result is due to D. Griffeath). If we could improve this result to $r_n^0 \to$ a.s. we would contradict the fact that r_n^0 returns to 0 infinitely often and prove $P(\Omega_\infty) = 0$. However we do not know how to do this and a simple example (see Brieman, 1968, page 58) shows that a random walk with independent increments may have $S_n = 0$ infinitely often and $S_n \to \infty$ in probability so we will not prove Griffeath's result here.

The last two paragraphs give the reasons that I believe $P(\Omega_\infty) = 0$ at p_c. To be fair to people with the opposite viewpoint (if there are any) I should note that the stationary measure μ constructed in (4.1) exists for $p = p_c$ and, by computations from the last section, has $E(r_1^\mu)^- \leq 1$ so near the right edge r_n^μ the density of particles does not go to 0. The last phenomenon might suggest $P(\Omega_\infty) > 0$ at p_c. Further explanation of this point will have to wait until Section 8 when we indicate the connection between $P(\xi_n^0 \neq \varnothing)$ and $P(0 \in \xi_n^Z)$.

6. Lower bounds on p_c. In this section we will use a branching process approximation and the characterization of p_c given in Section 3 to compute sequences of lower bounds for p_c. The first sequence of bounds increases very slowly to p_c but they are good, I think, for getting a feeling for why branching process methods do not work for oriented percolation and furthermore, we will see in the next section this sequence of bounds converges to p_c and allows us to reduce problems about percolation when $p < p_c$ to corresponding problems for subcritical branching processes.

The second set of bounds, based on (3.6) and (3.11) is less glamorous. They work better (the first is .5858, the ninth, according to Mauldon, 1960, is .6198) but they are not the last word. Using ideas of Gray, Smythe, and Wierman (1981) which have been developed further by Dhar (1982), one can conclude easily that $p_c \geq .618$ and with some work that $p_c \geq .6298$ (see their papers for details) so if you are only interested in the last word you can ignore the second part of this section.

The last lower bound compares favorably with numerical results:

.632	Kertesz and Vicsek (1980)
.644	Kinzel and Yeomans (1981)
.6445	Dhar and Barma (1981)
.6446	Blease (1977)

but unfortunately the best known upper bound $p_c \leq 0.84$ (given in Section 10) is much bigger than the lower bound so we do not rigorously know even the first digit in the decimal expansion of p. It is an important open problem to find a sequence of rigorous upper bounds which decrease to p_c.

Historically the first lower bounds on p_c were found by running the percolation process up to level k and using the distribution of the number of wet sites as the offspring distribution for the branching process. When the branching process dies out then the percolation process does. When $k = 1$ this gives $p_c \geq 1/2$. When $k = 2$ the occupation probabilities for the three sites on level 2 are p^2, $2p^2 - p^4$ and p^2 respectively (see Figure 5) so the mean $4p^2 - p^4$ is < 1 if

$$p^2 < (4 - \sqrt{12})/2$$

and it follows that

$$p_c \geq .5176.$$

When $k = 3$ the details are more complicated. The probability of reaching $(-3, 3)$ is still p^3 but there are 3 ways of reaching $(-1, 3)$ so if we let A_i be the probability the ith path is open and use the inclusion exclusion formula

$$P(A_1 \cup A_2 \cup A_3) = \textstyle\sum_i P(A_i) - \sum_{\{i,j\}} P(A_i \cap A_j) + P(A_1 \cap A_2 \cap A_3)$$

we see the probability of reaching $(-1, 3)$ is $3p^3 - 2p^5 - p^6 + p^8$ so the mean

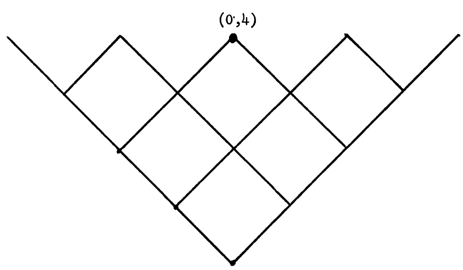

FIG. 5.

number of particles on level 3 is $8p^3 - 4p^5 - 2p^6 + 2p^8$. When $p = .531$ this is < 1 (and when $p_c = .532$ this is > 1) and we have

$$P_c \geq .531.$$

The computations for the $k \geq 4$ bounds are terrible and pointless. We might get $p_c \geq .545$ but the computations required are enormous. There are 6 paths from 0 to (0, 4), so there are $2^6 - 1$ combinations to consider and since the union of these 6 paths has 12 bonds the result will be a polynomial of degree 12.

A somewhat better sequence of bounds can be obtained by using the characterization of p_c given in Section 3. In that section we showed (see (3.6) and (3.11))

$$p_c = \sup\{p: \alpha(p) < 0\} \quad \text{where} \quad \alpha(p) = \inf_n E\bar{r}_n/n.$$

From this it follows that if

$$\pi_n = \sup\{p: E\bar{r}_n < 0\}$$

then

$$\sup_{n \leq N} \pi_n \uparrow p_c \quad \text{as} \quad N \to \infty.$$

REMARK. It is easy to see from the subadditivity and the trivial upper bound $\bar{r}_1 \leq 1$ that

$$\liminf_{m\to\infty} \pi_m \geq \sup_{n \leq N} \pi_n \quad \text{for all} \quad N$$

so $\pi_n \to p_c$ as $n \to \infty$. This is not important for applications because $\sup_{n \leq N} \pi_n$ converges to the desired limit and does so more rapidly.

It is easy to compute $E\bar{r}_1$. Just draw a picture (look at Figure 6) clearly $P(\bar{r}_1 = 1) = p$. If the bond from $(0, 0) \to (1, 1)$ is closed then $\bar{r}_1 = 1 - 2N$ where N is smallest integer $n \geq 1$ so that $(2n, 0) \to (-2n + 1, 1)$ or $(-2n + 2, 0) \to (-2n + 1, 1)$ is open. N is geometric with success probability $(2p - p^2)$ so $EN = 1/(2p - p^2)$ and

$$\begin{aligned} E\bar{r}_1 &= p \cdot 1 + (1 - p)E(1 - 2N) \\ &= 1 - (1 + p)\frac{2}{2p - p^2} = \frac{-2 + 4p - p^2}{2p - p^2}. \end{aligned}$$

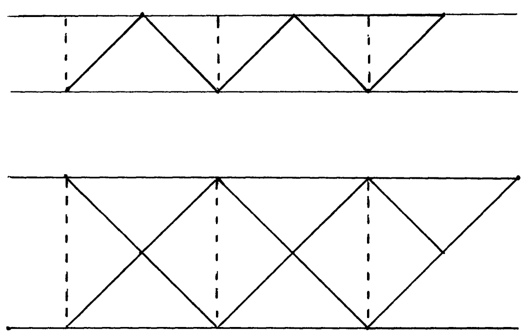

FIG. 6.

TABLE 1.

	r_c	$p_c = r_c \cos(\pi/4)$
1	.8284	.5858
2	.8442	.5970
3	.8538	
4	.8604	
5	.8652	
6	.8690	
7	.8720	
8	.8745	
9	.8767	.6198
∞	.8905	.6297

Solving we see $E\bar{r}_1 < 0$ if $p > 2 - \sqrt{2} = .5858$ so we have

$$p_c > .5858.$$

With a little more work you can compute $E\bar{r}_2$. The picture is more complicated (see Figure 6 again) but still breaks up into independent blocks. This time when we locate the first box which has an open path we have to compute the conditional probability of having an open path ending in the right upper arm of the X. We will not get involved in the details. Mauldon (1961) did computations which are equivalent to these and found the lower bounds for $k = 1, 2 \cdots 9$ shown in Table 1.

HISTORICAL REMARK. Mauldon (1961) introduced a reset process in which "every rth diagonal (and these alone) have the property that all the atoms on the diagonal between any two wet atoms are themselves wet." (Section 3, page 338). To compute the critical value of these processes he uses a result he proved earlier (see Mauldon, 1957) to conclude that if the right boundary has negative drift then the process dies out. In a remark in Section 10 of his paper he claims to prove that his sequence of bounds converges to the right limit. As we mentioned above, this result is correct. His proof, however, is not. All he shows is that if we let $\Omega^n = \{$the nth approximation wets infinitely many sites$\}$ then $P(\Omega^n) \to P(\Omega_\infty)$ as $n \to \infty$. If $\pi_n = \sup\{p: P(\Omega^n) = 0\}$ then this certainly implies

$$\lim \sup_{n\to\infty} \pi_n \le p_c$$

but just knowing $P(A_n) \to P(\Omega_\infty)$ or even $P(A_n) \downarrow P(\Omega_\infty)$, it is not possible to conclude that $\sup\{p: P(A_n) = 0\} \to p_c$ for if we let $A_n = \{\xi_n \ne \varnothing\}$ then $\sup\{p: P(A_n) = 0\} = 0$ for all n but $p_c > 1/2$.

7. Exponential estimates for $p < p_c$. In this section we will prove some results which show that in the subcritical case ξ_n^0 dies out exponentially fast and that $\bar{r}_n \to -\infty$ exponentially fast. The estimates (1), (2), (4), (5), and (7) below are due to Griffeath (1981). Only (3), (6) and the existence of $\lim(1/n) \log P(\xi_n^0 \ne \varnothing)$ are new.

Here and in Sections 10–13 we will be dealing with exponential estimates so to simplify computations we make the following.

ANNOUNCEMENT. The C and γ below are numbers in $(0, \infty)$ whose values may change from line to line.

I hope the reader will agree that this notational convention allows us to dispense with such tedious tasks as explicitly summing geometric series or estimating the maximum of a bounded function. In any case we will indulge in this luxury below.

(1) If $p < p_c$ then there is a constant $\gamma > 0$ (which depends on p) so that

$$P(\xi_n^0 \neq \varnothing) \leq e^{-\gamma n}$$

and $(1/n) \log P(\xi_n^0 \neq \varnothing) \to \gamma$ as $n \to \infty$.

PROOF. When $\xi_n^0 \neq \varnothing$, $|\xi_n^0| \geq 1$ so it is clear that

$$P(\xi_{m+n}^0 \neq \varnothing) \leq P(\xi_m^0 \neq \varnothing) P(\xi_n^0 \neq \varnothing).$$

If we let $a_n = \log P(\xi_n^0 \neq \varnothing)$ then the last inequality implies

$$a_{m+n} \geq a_m + a_n$$

i.e. a_m is a superadditive sequence. It is easy to see that the last inequality implies

$$\lim \inf_{n\to\infty} a_n/n \geq a_m/m$$

for all m so we have

$$\lim_{n\to\infty} a_n/n = \sup_{m\geq 1} a_m/m$$

and it follows that

$$a_n/n \leq \sup_{m\geq 1} a_m/m \equiv -\gamma$$

so we have proved everything except that $\gamma > 0$. To do this we will, of course, have to use the fact that $p < p_c$. Up to this point everything holds for any $p \leq 1$.

If $p < p_c$, then by results in Section 2 there is an N so that $E\bar{r}_N < 0$ and from subadditivity it follows that

$$\bar{r}_{0,mN} \leq \bar{r}_{0,N} + \bar{r}_{N,2N} + \cdots + \bar{r}_{(m-1)N,mN}.$$

The right hand side (which we call S_m) is a random walk with $ES_1 < 0$ and $S_1 \leq N$ so $\varphi(\theta) = E \exp(\theta S_1) < \infty$ for all $\theta \geq 0$. If we pick M large and let $\bar{\varphi}(\theta) = E \exp(\theta(S_1 \vee -M))$ then

$$\lim \sup_{\theta\to 0} \frac{\varphi(\theta) - 1}{\theta} \leq \lim_{\theta\to 0} \frac{\bar{\varphi}(\theta) - 1}{\theta} = E(S_1 \vee -M).$$

Letting $M \to \infty$ it follows that

$$\lim \sup_{\theta\to 0} \frac{\varphi(\theta) - 1}{\theta} \leq ES_1 < 0$$

so we can pick $\theta_0 > 0$ with $\varphi(\theta_0) < 1$. But then

$$P(S_m \ge 0) \le E \exp(\theta_0 S_m) = \varphi(\theta_0)^m$$

so $P(\bar{r}_{mN} \ge 0) \to 0$ exponentially fast. To conclude that the same thing is true for $P(\xi_n \ne \varnothing)$ observe that

$$P(\xi_n = \varnothing) \ge P(\bar{r}_n < 0 < \bar{\ell}_n)$$

so

$$P(\xi_n \ne \varnothing) \le P(\bar{r}_n \ge 0) + P(\bar{\ell}_n \le 0) = 2P(\bar{r}_n \ge 0)$$

and the proof of (1) is complete.

REMARK. From the proof above it follows that

(2) if $b > \alpha$ then there are constants C, γ so that

$$P(\bar{r}_n > bn) \le Ce^{-\gamma n}.$$

We will consider the more delicate problem of studying the deviations $P(\bar{r}_n \le an)$ for $a < \alpha$ in Section 11.

From (1) it follows immediately that if $p < p_c$ then $\bar{r}_n \to -\infty$ exponentially fast. We will prove this in a minute and identify the rate. To state this result we have to make:

ANNOUNCEMENT 2. The γ's are numbered by the formula in which they are defined. For example, the γ in (1) above will be called γ_1 in this section and in other sections we will refer to it as $\gamma_{7.1}$. When, as in (2) above, γ is not given explicitly we define it to be the sup of all the numbers γ for which there is a C which makes the inequality valid. (In general the inequality will not hold for this value but this convention is convenient for Section 14.)

Using the convention described in the first half of the announcement we can state our result for $\bar{r}$.

(3) if $p < p_c$ then

$$(1/n)\log(-\bar{r}_n) \to \gamma_1 \quad \text{a.s.}$$

REMARK. This trivially implies $\alpha(p) = -\infty$ for $p < p_c$. The conclusion given here is a small improvement of the original result of Griffeath (1981).

$$\lim \inf_{n\to\infty}(1/n)\log(-\bar{r}_n) \ge \varepsilon.$$

The author would like to thank J. Neveu who suggested that the sharper result should be true.

PROOF. Let $c_n = 1/P(\xi_n^0 \ne \varnothing)$ and let $U_n = \sup\{x \le 0 : \xi_n^x \ne \varnothing\}$.

$$P\left(U_n > \frac{-c_n}{n^2}\right) \le \frac{1}{2}\,\frac{c_n}{n^2}\,P(\xi_n^0 \ne \varnothing) = \frac{1}{2n^2}$$

so the Borel Cantelli lemma implies

$$P(U_n > -c_n/n^2 \text{ i.o.}) = 0.$$

Let $V_n = \sup\{x \le 0 : \xi_n^{2nx} \ne \varnothing\}$. The events $\{\xi_n^{2nx} \ne \varnothing\}$ are independent so

$$P\left(V_n < \frac{-nc_n}{2}\right) = \left(1 - \frac{1}{c_n}\right)^{nc_n/2}.$$

Now as $n \to \infty$ $(1 - 1/c_n)^{c_n/2} \to e^{-1/2}$ so if n is large

$$P\left(V_n < -\frac{nc_n}{2}\right) \le e^{-n/4}.$$

Since $U_n \ge 2nV_n$ it follows from the Borel Cantelli lemma that

$$P(U_n < -n^2 c_n \text{ i.o.}) = 0.$$

Combining this with the other result about U_n we see that with probability 1

$$|\log(-U_n) - \log c_n| \le 2 \log n$$

for all n sufficiently large. Since $U_n - n \le \bar{r}_n \le U_n + n$ and $(1/n) \log c_n \to \gamma_1$, it follows that $(1/n) \log(-\bar{r}_n) \to \gamma_1$.

Another Corollary of (1) is

(4) $$\text{if} \quad p < p_c \quad \text{then} \quad E\,|C_0| < \infty.$$

Proof. If $\xi_n = \varnothing$, $|C_0| \le 1 + 2 + \cdots + (n-1) = n(n+1)/2 \le n^2$(if $n \ge 1$) so

$$P(|C_0| > n^2) \le P(\xi_n \ne \varnothing) \le e^{-\lambda_1 n}$$

and it follows that

$$E\,|C_0| = \textstyle\sum_{m=1}^{\infty} P(|C_0| \ge m) < \infty.$$

The argument above leads to the bound

(5) $$P(|C_0| \ge m) \le \exp(-\gamma_1 m^{1/2}).$$

By working harder it is possible to improve the result to

(6) if $p < p_c$ there is a constant $\gamma > 0$ so that

$$P(|C_0| \ge m) \le e^{-\gamma m}$$

and $(1/m) \log P(|C_0| \ge m) \to -\gamma$ as $m \to \infty$.

Remark. There are two reasons we are interested in this result:

(i) if $a > \gamma_1/\gamma_6$

$$P(|C_0| > an \mid \xi_n^0 \ne \varnothing) \le \frac{P(|C_0| > an)}{P(\xi_n^0 \ne \varnothing)} \le \exp(-\gamma_6 an + \gamma_1 n)$$

which converges to 0 exponentially fast so when $\xi_n^0 \ne \varnothing$ the cluster is long but narrow—with high probability it's average width is no more than γ_1/γ_6.

(ii) if we consider the supercritical analogue of (1) and (6) then we have

$$P(n < \tau^0 < \infty) \le Ce^{-\gamma n}$$

but

$$ce^{-\Gamma n^{1/2}} \le P(n \le |C_0| < \infty) \le Ce^{-\gamma n^{1/2}}.$$

See Section 12.

Proof of (6). As the reader can probably guess from the statement, subadditivity is at work. If T is the first time $\sum_{n=0}^{T} |\xi_n^0| \ge m$ then we must have $|\xi_T^0| \ge 1$. It follows from the Markov property that

$$P(|C_0| \ge m + n) \ge P(|C_0| \ge m)P(C|C_0| \ge n)$$

which as in (1) implies everything but the positivity of the constant (which *never* comes for "free" since the first part of the argument does not depend on the size of p.)

To prove that the constant is positive we observe

(7) If $p < p_c$ then as $n \to \infty$

$$E(|\xi_n^0|) \le (n + 1)e^{-\gamma_1 n} \to 0.$$

(7) implies that if we pick N large enough then $E|\xi_N^0| \le 1$. If we remember the branching process approximation of the last section, we see that to prove (6), it suffices to prove that the conclusion is valid for a branching process in which the mean of the offspring distribution < 1 and the total number of offspring is $\le N + 1$. This will give us a bound on $\sum_{m=0}^{\infty} |\xi_{mN}^0|$ and a trivial comparison shows

$$\sum_{n=0}^{\infty} |\xi_n^0| \le N^2 \sum_{m=0}^{\infty} |\xi_{mN}^0|.$$

To prove the result for the branching process, we use a formula for the total progeny which we learned from P. Jagers and O. Nerman. The observation is due to Papangelou but the trivial proof below is due to L. Bondeson. Let $X_1, X_2 \cdots$ be independent random variables with $P(X_m = j) = P(|\xi_N^0| = j)$ and define a random walk S_n by $S_0 = 1$ and for $n \ge 1$, $S_n = S_{n-1} + (X_n - 1)$. If we modify the dynamics of the Galton-Watson process so that each time *one* (and only one) of the individuals alive is chosen to die and at death gives birth to j individuals with probability $P(X_m = j)$, then S_n is the number of individuals alive at time n in the modified process.

Since the individuals in a branching process reproduce independently, the total progeny of the branching process has the same distribution as the total progeny of the modified process. Only 1 person dies at each instant in the modified process. If we let $\tau = \inf\{n: S_n = 0\}$ then the total number of individuals who have ever lived (including the individual who started the process) is τ. Since

S_n is a random walk with $S_n - S_{n-1} \leq M$, it follows from the second part of the proof of (1) that if $ES_1 < 0$ then there are constants C, γ so that

$$P(\tau \geq n) \leq P(S_n > 0) \leq Ce^{-\gamma n}$$

and the proof of (6) is complete.

REMARK. The proof of (6) shows us that "anything that is true for a subcritical branching process is true for subcritical oriented percolation". In Section 9 we give the supercritical analogue: "anything that is true for p sufficiently close to 1 is true for all $p > p_c$".

8. Time reversal duality, first results for $p > p_c$. If $p > p_c$ then the results of Section 3 imply that on Ω_∞, $\ell_n/n \to -\alpha$ and $r_n/n \to \alpha$. In this section we will investigate what happens between ℓ_n and r_n. The first step is to observe that (3.10c) implies

(1)
$$\xi_n^0 = \xi_n^Z \cap [\ell_n, r_n].$$

TECHNICAL NOTE. Here we use ξ_n^Z as a replacement for the cumbersome $\xi_n^{(-\infty,\infty)}$ of Section 3. Actually $\xi_n^Z = \xi_n^{2Z}$ (recall the definition of ξ_n^A in Section 3) but the two is a nuisance.

(1) tells us that in the interval $[\ell_n, r_n]$ ξ_n^0 looks like ξ_n^Z. The key to studying ξ_n^Z is the following duality equation:

(2) If n is an even integer

$$P(\xi_n^A \cap B \neq \varnothing) = P(\xi_n^B \cap A \neq \varnothing).$$

PROOF. $\{\xi_n^A \cap B \neq \varnothing\} = \{$there are $x \in A$ and $y \in B$ so that $(x, 0) \to (y, n)\}$. If we map $\mathscr{L}$ into itself by sending $(z_1, z_2) \to (z_1, n - z_2)$ and reverse the orientation of the bonds, the distribution of the process is unchanged but the event under consideration becomes $\{$there are $y \in B$ and $x \in A$ so that $(y, 0) \to (x, n)\} = \{\xi_n^B \cap A \neq \varnothing\}$.

REMARK. There is obviously a duality equation for odd integers n but it is awkward to state since we have the even integers at one end and the odd at the other. To simplify things then we will avoid this formula and consider only what happens to ξ_{2n}^Z as $n \to \infty$.

If we let $A = Z$ in (2) we get

(3)
$$P(\xi_{2n}^Z \cap B \neq \varnothing) = P(\xi_{2n}^B \neq \varnothing).$$

Letting $n \to \infty$ in (3) we see that

(4)
$$P(\xi_{2n}^Z \cap B \neq \varnothing) \downarrow P(\tau^B = \infty).$$

Introducing coordinate notation

$$\xi_n^Z(x) = |\xi_n^Z \cap \{x\}| = \begin{cases} 1 & \text{if } x \in \xi_n^Z \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for all k and $\{x_1, \cdots x_k\} \subset 2\mathbb{Z}$ the finite dimensional distributions $P(\xi_{2n}^Z(x_i) = y_i,\ i = 1, \cdots k)$ converge to those of a limit which we denote by $\xi_\infty^Z(x)$, $x \in 2\mathbb{Z}$.

If we consider $\xi_\infty^Z(x)$, $x \in 2\mathbb{Z}$, as a process indexed by $x \in 2\mathbb{Z}$ then it is a stationary sequence. In fact

$$(5) \qquad \xi_\infty^Z(x) \quad \text{is ergodic.}$$

PROOF. Let $\eta_n = \{x\text{:there is a } y \text{ so that } (y, -n) \to (x, 0)\}$ and let $\eta_\infty = \bigcap_{n=1}^\infty \eta_n$. If we let $\eta_n(x) = |\eta_n \cap \{x\}|$ for $1 \le n \le \infty$ then we have $\eta_1(x) \ge \eta_2(x) \ge \cdots$ and $\eta_n(x) \downarrow \eta_\infty(x)$ as $n \uparrow \infty$. For even n $\eta_n(x)$ has the same finite dimensional distributions as $\xi_n^Z(x)$ so this is true also for $n = \infty$. Let f be a function on $\{0, 1\}^k$ which is increasing in the partial ordering on this space i.e. if $\eta_i \le \zeta_i$, $i = 1, \cdots, k$ then $f(\eta) \le f(\zeta)$. For such an f

$$(6) \qquad f(\eta_n(2x+2), \cdots, \eta_n(2x+2k)) \ge f(\eta_\infty(2x+2), \cdots, \eta_\infty(2x+2k))$$

for all n, $x \in Z$. Now the random variables $\eta_n(x)$ $x \in 2Z$ have the property that if $|x - y| \ge 2n$ $\eta_n(x)$ is independent of $\eta_n(y)$ so it follows from the strong law of large numbers that

$$\frac{1}{2N+1} \sum_{x=-N}^N f(\eta_n(2x+2), \cdots, \eta_n(2x+2k)) \to Ef(\eta_n(2), \cdots, \eta_n(2k))$$

(break the sum into $n + k$ pieces each of which is a sum of independent random variables). Combining the last result with (6) and letting $n \to \infty$ gives

$$\limsup_{N\to\infty} (1/(2N+1)) \sum_{x=-N}^N f(\eta_\infty(2x+2), \cdots, \eta_\infty(2x+2k)) \le Ef(\eta_\infty(2), \cdots, \eta_\infty(2k))$$

and applying the ergodic theorem it follows that if we let $\mathscr{I}$ denote the σ-field of shift invariant events

$$E(f(\eta_\infty(2), \cdots, \eta_\infty(2k)) \mid \mathscr{I}) \le E(f(\eta_\infty(2), \cdots, \eta_\infty(2k))).$$

Now the number on the right is the mean of the random variable on the left so there cannot be strict inequality on a set of positive measure, i.e.

$$(7) \qquad E(f(\eta_\infty(2), \cdots, \eta_\infty(2k)) \mid \mathscr{I}) = E(f(\eta_\infty(2), \cdots, \eta_\infty(2k))).$$

At this point we have shown that (7) holds for increasing functions. Since every function on $\{0, 1\}^k$ is a difference of two increasing functions, it follows that (7) holds for any function of finitely many sites. Taking limits and using the inequality $E|E(f \mid \mathscr{I}) - E(g \mid \mathscr{I})| \le EE(|f - g| \mid \mathscr{I}) = E|f - g|$ it follows that (7) holds for any bounded f. This implies $\mathscr{I}$ is trivial and completes the proof.

From (4) and (5) it follows immediately that we have

(8) $$\frac{1}{2N+1}\sum_{x=-N}^{N} \xi_\infty^Z(x) \to P(0 \in \xi_\infty^Z) = P(\Omega_\infty)$$

and consequently that if $P(\Omega_\infty) > 0$ we have

(9) $$P(\xi_\infty^Z(x) \equiv 0) = 0$$

(10) $$P(\xi_\infty^Z \cap [-N, N] = \varnothing) \downarrow 0 \quad \text{as} \quad N \uparrow \infty$$

and hence that

(11) $$P(\xi_n^{[-N,N]} \neq \varnothing \text{ for all } n) \uparrow 1 \quad \text{as} \quad N \uparrow \infty$$

a result which is obvious but seems difficult to prove directly.

The last result is an ingredient for the construction to be described in the next section. Another fact that will be useful there and at several other points below is

(12) $\xi_{2n}^Z(x)$, $x \in 2Z$, has positive correlations in the sense of (5.6): If f and g are nonnegative increasing functions

$$Ef(\xi_{2n}^Z)g(\xi_{2n}^Z) \geq Ef(\xi_{2n}^Z)Eg(\xi_{2n}^Z).$$

PROOF. If f and g depend upon finitely many coordinates then $f(\xi_{2n}^Z)$ is and $g(\xi_{2n}^Z)$ are increasing functions of a finite number of independent 0, 1 valued random variables, so the result is a consequence of (5.6). The general result follows by taking limits to conclude first that the result is true for bounded measurable f, g and then using the monotone convergence theorem.

9. A construction for studying $p > p_c$ (and showing $\alpha(p_c) = 0$). In this section we will introduce a construction which will allow us to reduce questions about oriented percolation with $p > p_c$ to corresponding questions about a 1-dependent site percolation process with p arbitrarily close to 1, a situation in which it is easy to prove the desired results.

The first thing to do is to define the site percolation process and describe its relationship to the original process. Let $\mathscr{L} = \{(m, n): m + n \text{ is even}, n \geq 0\}$. With each $z \in \mathscr{L}$ there is associated a random variable $\eta(z)$ such that $\eta(z) = 1$ (i.e. z is open) with probability π and $\eta(z) = 0$ with probability $1 - \pi$. To define the $\eta(z)$ we will pick δ small, L large, and define for each $(m, n) \in \mathscr{L}$

$$C_{m,n} = ((1-\delta)\alpha Lm, Ln)$$

$$R_{m,n} = C_{m,n} + [(-1-\delta)\alpha L, (1+\delta)\alpha L] \times [0, (1+\delta)L]$$

where α is the constant defined in Section 3. We will set $\eta(m, n) = 1$ if a certain good event $G_{m,n}$ occurs in $R_{m,n}$.

Before going into the details of what the good event is, I will first give its three crucial properties:

(i) if $\delta \leq .1$ the random variables $\eta(z)$ will be 1-dependent, that is, if we let $\|(m, n)\| = (|m| + |n|)/2$ and if $z_1, \cdots z_m$ are points with $\|z_i - z_1\| > 1$ for $i \neq j$ then $\eta(z_1), \cdots, \eta(z_m)$ are independent.

(ii) if percolation occurs in the η-system then there is an infinite path in the original system which starts in $[-1.5\delta\alpha L, 1.5\delta\alpha L]$.

(iii) if δ, $\varepsilon > 0$ and $p > p_c$ then we can pick L large enough so that $P(\eta(z) = 1) > 1 - \varepsilon$.

Having announced our aims, the next step is to define the good event and show it has the desired properties. Let A_{00} be the parallelogram with vertices

$$u_0 = (-1.5\delta\alpha L, 0) \qquad v_0 = (-.5\delta\alpha L, 0)$$

$$u_1 = u_0 + (1 + \delta)(\alpha L, L) \qquad v_1 = v_0 + (1 + \delta)(\alpha L, L)$$

and let $B_{00} = -A_{00}$. We say that G_{00} occurs if:

(i) there is a path from $[u_0, v_0]$ to $[u_1, v_1]$ which stays in A_{00} and (ii) there is a path from $[-v_0, -u_0]$ to $[-v_1, -u_1]$ which stays in B_{00}. See Figure 7 for a picture.

The events $G_{m,n}$ are defined by translating the last definition by $C_{m,n}$. (Note: For this to work exactly we need $(1 - \delta)\alpha L$ and L to be even integers. Since the construction in our proof is already complicated enough, we will ignore annoying little details like this).

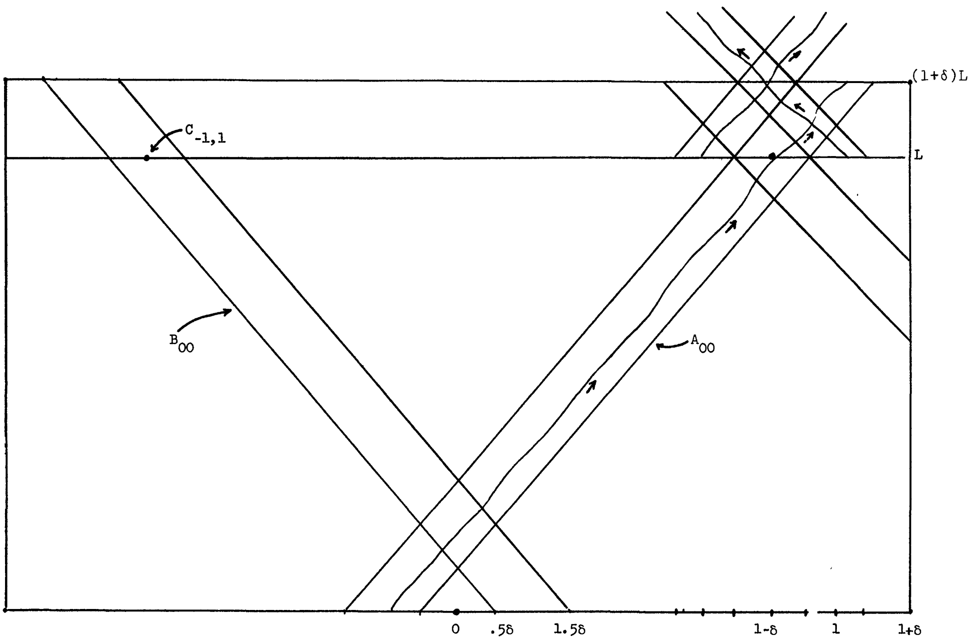

FIG. 7. The numbers on the bottom scale are multiples of αL.

NOTE. To prepare for the proof of $\alpha(p_c) = 0$ the reader should check as he goes along that the proof given below only uses the fact that $\alpha(p) > 0$.

Property (i) is obvious. To see this one observes that $G_{0,0}$ just depends on the bonds in $R_{0,0}$ and one checks that $R_{0,0}$ intersects $R_{2,0}$ and $R_{1,1}$ but not $R_{4,0}$, $R_{3,1}$ or any $R_{m,2}$.

Property (ii) is also easy to check. To see this consult Figure 7 and follow the arrows to see that on $G_{0,0} \cap G_{1,1}$ there is a path from $[-1.5\delta\alpha L, 1.5\delta\alpha L] \times \{0\}$ through $C_{1,1} + ((-.5\delta\alpha L, .5\delta\alpha L) \times \{0\})$ and on up to $C_{0,2} + ((-.5\delta\alpha L, .5\delta\alpha L) \times \{0\})$ and to $C_{2,2} + ((-.5\delta\alpha L, .5\delta\alpha L) \times \{0\})$. From this observation and induction it follows easily that if there is an infinite path in the η-system then there is a corresponding infinite path in the original system (but *not* conversely).

Last but not least we have to check (iii). Let $\hat{r}_n = \sup \xi_n^{(-\infty,-.8\delta\alpha L]}$ and let $\bar{r}_n = \sup \xi_n^{(-\infty,0]}$. $\{\hat{r}_n + .8\delta\alpha L, n \geq 0\} =_d \{\bar{r}_n : n \geq 0\}$ and as $n \to \infty$, $\bar{r}_n/n \to \alpha$ a.s. so it follows that if we pick L large enough then with probability $\geq 1 - \varepsilon/4$ we have

$$\hat{r}_{(1+\delta)L} > -.8\delta\alpha L + (1 + .9\delta)\alpha L$$

and for $n \leq (1 + \delta)L$

$$\hat{r}_n \leq -.7\delta\alpha L + n\left(\frac{1 + 1.1\delta}{1 + \delta}\right)\alpha.$$

The last two events guarantee that there is a path from $(-\infty, -.8\delta\alpha L] \times \{0\}$ up to $[(1 + .1\delta)\alpha L, (1 + .4\delta)\alpha L] \times \{(1 + \delta)L\}$ which does not cross the line $[v_0, v_1]$.

To prove that this path does not fall too far to the left we observe that to travel from the line $[u_0, u_1]$ to $[\alpha L, \infty) \times \{(1 + \delta)L\}$ a path must have an average slope of at least $b = \alpha(1 + 1.5\delta)/(1 + \delta) > \alpha$ and it follows from (7.2) that

$$P(\bar{r}_n > bn) \leq Ce^{-\gamma n}$$

so picking M large enough so that

$$\sum_{n=M}^{\infty} Ce^{-\gamma n} \leq \varepsilon/8$$

and then considering separately the points on $[u_0, u_1]$ with $y \leq (1 + \delta)L - M$ and $y > (1 + \delta)L - M$ we see that if L is large the probability that there is a path connecting $[u_0, u_1]$ and $[\alpha L, \infty)$ is $\leq \varepsilon/4$.

Combining the results of the last two paragraphs we see that if L is sufficiently large then with probability $\geq 1 - \varepsilon/2$ the first half of the good event occurs. Since the second half of the good event has the same probability as the first it follows that with probability $\geq 1 - \varepsilon$ the good event occurs and we have shown (iii).

Having struggled to complete the construction we can now reap the benefits. The easiest (and historically the first) consequence is

$$(1) \qquad \alpha(p_c) = 0.$$

PROOF. To prove this result we need to use one simple fact which we will prove in the next section: if $P(\eta(z) = 1) > 1 - 3^{-36}$ then there is positive probability of percolation in the η-system. Taking this fact for granted, the rest is easy. If $\alpha(p_c) > 0$ let $\delta = .1$ and pick L so large that $P(\eta(z) = 1) > 1 - 3^{-37}$. There are only a finite number of bonds in $R_{0,0}$ so we can pick $p < p_c$ so that $P(\eta(z) = 1) > 1 - 3^{-36}$ but this implies that there is positive probability of percolation when the parameter value is p, a contradiction.

REMARK. The construction in this section was inspired by an argument of Russo (1981). The version given above is, thanks to Larry Gray, considerably simpler than the proof given in Durrett and Griffeath (1983).

10. $P(\tau^A < \infty) \le C \exp(-\gamma\,|A|)$. In this section we will prove the exponential estimate given in the title ($\tau^A = \inf\{n: \xi_n^A = \varnothing\}$). We will prove the result first for intervals and then use a comparison to prove the result holds in general. To prove the result for intervals we will follow the plan alluded to in Section 9: we will prove the result first for 1 dependent site percolation with p close to 1 and then use the construction to conclude that the result holds for all $p > p_c$. The same argument allows one to conclude that $p_c \le 8/9$.

Consider site percolation on $\mathscr{L} = \{(m, n): m + n \text{ is even and } n \ge 0\}$. Let $A = \{-2N, \cdots 0\}$. Let $C = \{z$: there is a $y \in A$ with $(y, 0) \to z\}$. Let

$$D = \{(a, b) \in R^2 : |a| + |b| \le 1\}$$

and let

$$W = \cup_{z \in C}(z + D).$$

If $|C| < \infty$ let Γ be the boundary of the unbounded component of $(R \times (-1, \infty)) - W$ and orient the boundary in such a way that the segment from $(0, -1)$ to $(1, 0)$ (which is always present) is oriented in the direction indicated. Γ is the contour associated with the cluster $\mathscr{L}$ (see Figure 8 for a picture). The idea of the contour method is to estimate

$$P(|C| < \infty) = P(\Gamma \text{ exists}) \le E(\text{no. of contours}).$$

Although the details are tedious to write down, the "contour argument" is very simple. Since the contour never passes through an arc twice, there are at most 3^{m-1} contours of length m. (The first segment is always $(0, -1) \to (1, 0)$ and after that there are at most three choices at each stage.) On the other hand for a contour of length m to exist there must be at least $m/4$ sites which are closed (we will show this below). Taking into account that the shortest possible contour has length $2N + 4$, it follows that if the sites are independent

$$P(\tau^{[-2N,0]} < \infty) \le \sum_{m=2N+4}^{\infty} 3^m (1-p)^{m/4} = C(3(1-p)^{1/4})^{2N} \tag{1}$$

if $1 - p < 3^{-4}$ so $p_c \le 80/81$ for (independent) site percolation. This is not a very good bound but at least it shows $p_c < 1$. To handle the 1-dependent case we observe that there are 9 sites in $\mathscr{L}$ with $\|(m, n)\| = (|m| + |n|)/2 \le 1$ so for each Γ of length m there is a set of $m/36$ sites which are separated by more than

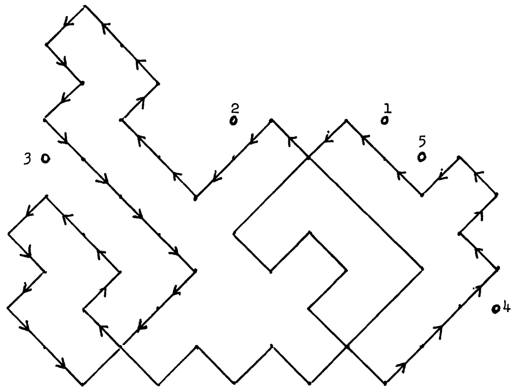

FIG. 8.

1 and which must be closed for the contour to exist. Therefore if $1 - p < 3^{-36}$,

$$P(\tau^{[-2N,0]} < \infty) \le C(3(1-p)^{1/36})^{2N}. \tag{2}$$

To complete the proofs of (1) and (2) we need to show that outside any contour of length m there are $m/4$ sites which must be closed for the contour to exist. To do this requires a series of definitions and observations:

(a) A segment is a line segment of the form $x + F$ where $x \in C$ and F is one of the sides of D. If we stand at the midpoint of one of the segments which makes up Γ and face in the direction of the orientation then our left hand is in W and our right hand is in W^c. The site closest to our right hand is called the site associated with the segment.

(b) We call segments of Γ which look like ↖ ↙ ↘ and ↗ segments of types 1, 2, 3, and 4 (respectively). A look at the sites labeled 1, 2, 3, and 4 in Figure 8 will convince you that a site associated with a segment of type 1 or 2 must be closed but a site associated with a segment of type 3 or 4 may be open or closed.

(c) At this point it should be clear that the 1's and 2's are our friends and the 3's and 4's are our enemies. Fortunately the former are more numerous. 1's and 2's decrease our x coordinate by 1, and 3's and 4's increase it by 1. If we let m_i be the number of segments of type i, then since the contour starts at $(0, -1)$ and ends at $(-2N, -1)$, $m_1 + m_2 = m_3 + m_4 + 2N$. Hence if the contour has length m, $m_1 + m_2 \ge m/2$.

(d) Unfortunately a site in W^c can be associated with two boundary segments (look at 5 in Figure 9). Since each site is associated with at most one segment of each type, the number of sites associated with segments of types 1 and 2 is $\ge (m_1 + m_2)/2 \ge m/4$, and we have proved the desired results.

REMARK. If we are dealing with bond percolation then each segment in Γ of type 1 and 2 is associated with 1 closed bond (the one which crosses it), so we get $p_c \le 8/9$. This estimate can be improved by a more careful counting of contours. If we observe that it is impossible for a 3 to be followed by a 4 or for a 4 to be

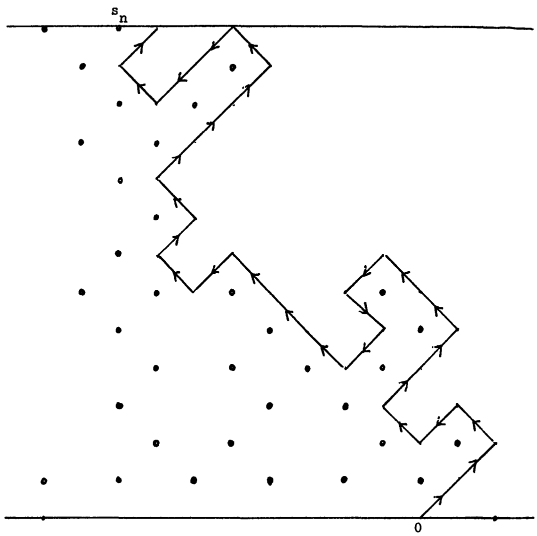

FIG. 9.

followed by a 3 we see the numbers of contours of length $n \le (A^n)_{43}$ where

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The largest eigenvalue of this matrix is $(3 + \sqrt{5})/2 = 2.62$ so $p_c \le 1 - (2.62)^{-2} \approx .84$. By working harder it is possible to get slightly better estimates, but the details become very complicated and the effort is pointless since it is not known how to find upper bounds which decrease to p_c.

Having proved (2) it is a simple matter to prove the corresponding result for oriented percolation.

(3) If $p > p_c$ there is a constant γ so that $P(\tau^{(-2n,-2]} < \infty) \le e-\gamma n$ and

$$\lim_{n\to\infty} 1/n \log P(\tau^{(-2n,-2]} < \infty) = -\gamma.$$

PROOF. Since $\xi_n^{A\cup B} = \varnothing$ if and only if $\xi_n^A = \varnothing$ and $\xi_n^B = \varnothing$ we have

$$P(\xi_n^{A\cup B} = \varnothing) = P(\xi_n^A = \varnothing, \xi_n^B = \varnothing) \ge P(\xi_n^A = \varnothing)P(\xi_n^B = \varnothing)$$

by Harris' inequality (5.6). Letting $n \to \infty$ gives

$$P(\tau^{A\cup B} < \infty) \ge P(\tau^A < \infty)P(\tau^B < \infty).$$

If we let $a_n = \log P(\tau^{(-2n,-2]} < \infty)$ then $a_{m+n} \ge a_m + a_n$, and repeating the proof of (7.1) proves everything but the positivity of the constant. To do this we use the construction in Section 9. Let $\delta = .1$ and pick L large enough so that $P(\eta(z) = 1) > 1 - 3^{-36}$. Starting with $[-1.1\alpha L - 1.8\alpha LN, 1.1\alpha L]$ in the original process corresponds to starting with $[-2N, 0]$ in the η-process. (To make the analogy exact, place the first interval at a height L above the y-axis.) It follows from (2) that if $p > p_c$ there are constants c, γ so that

$$P(\tau^{[-2M,0]} < \infty) \le Ce^{-\gamma M}.$$

At this point we have proved the inequality for intervals. To extend the result to arbitrary sets A we will prove a comparison result. We say that A is more spread out than B (and write $A \succ B$) if there is an increasing function φ from B into A so that for all $x, y \in B$ $|\varphi(x) - \varphi(y)| \ge |x - y|$. (Note that this implies φ is $1 - 1$ so $|A| \ge |B|$.)

(4) If $A \succ B$ then

$$P(\tau^A < \infty) \le P(\tau^B < \infty).$$

Since any set is more spread out than an interval with the same number of points, (4) shows that

$$P(\tau^A < \infty) \le \exp(-\gamma_3 |A|). \tag{5}$$

PROOF OF (4). We will construct the two systems ξ_n^A and ξ_n^B on the same probability space in such a way that $\xi_n^A \succ \xi_n^B$ for all n. To do this we start with two copies of $\mathscr{L}$ ($\mathscr{L}_1$ and $\mathscr{L}_2$) and a collection of i.i.d. random variables which determine whether the bonds in the first lattice are open. We define the system ξ_n^B in the usual way on $\mathscr{L}_1$. We will build up ξ_n^A on $\mathscr{L}_2$ row by row. Assume that $\xi_n^A \succ \xi_n^B$ (this holds when $n = 0$) and let φ_n be a map from ξ_n^B into ξ_n^A which has $|\varphi_n(x) - \varphi_n(y)| \ge |x - y|$ $x, y \in \xi_n^B$. Let the bonds $(\varphi_n(x), n) \to (\varphi_n(x) \pm 1, n + 1)$ in $\mathscr{L}_2$ have the same state (i.e. open or closed) as the corresponding bonds $(x, n) \to (x \pm 1, n + 1)$ in $\mathscr{L}_1$ and use independent random variables to determine the state of the rest of the bonds in $\mathscr{L}_2$ which lie in $R \times (n, n + 1)$.

I claim that $\xi_{n+1}^A \succ \xi_{n+1}^B$. To prove this let $y \in \xi_{n+1}^B$, let

$$\psi(y) = \inf\{x\colon x \in \xi_n^B \text{ and } (x, n) \to (y, n + 1)\} \quad (\psi(y) = y - 1 \text{ or } y + 1),$$

and let

$$\varphi_{n+1}(y) = \varphi_n(\psi(y)) + (y - \psi(y)).$$

$(\psi(y), n) \to (y, n + 1)$ is open so it follows from the coupling that $(\varphi_n(\psi(y)), n) \to (\varphi_{n+1}(y), n + 1)$ is also. Hence $\varphi_{n+1}(y) \in \xi_{n+1}^A$. On the other hand if $y < z$ are in ξ_{n+1}^B then $\psi(y) \le \psi(z)$ so $\varphi_n(\varphi(z)) - \varphi_n(\psi(y)) \ge \psi(z) - \psi(y)$. We then have

$$\varphi_{n+1}(z) - \varphi_{n+1}(y) = \varphi_n(\psi(z)) - \varphi_n(\psi(y)) + (z - \psi(z)) - (y - \psi(y)) \ge z - y$$

which completes the proof.

REMARK. The reader should note that (4) gives another sense in which ξ_n^A is a monotone function of A. Since this notion is invariant under translation, $\bar{\xi}_n - \bar{r}_n$ (see Section 4 for the notation) is a decreasing sequence with respect to $\prec$. Unfortunately this does not imply that the sequence converges.

11. Large deviations results for $\bar{r}_n$. In this section we will prove

(1) If $p > p_c$ and $a < \alpha(p)$ there is a constant $\gamma(a)$ such that

$$P(\bar{r}_n \le an) \le e^{-\gamma(a)n} \quad \text{and} \quad \lim_{n\to\infty}(1/n)\log P(\bar{r}_n \le an) = -\gamma(a).$$

PROOF. After (7.1), (7.6), and (10.3) you should be able to guess how the first part of the argument goes. One observes that

$$P(\bar{r}_{m+n} \le a(m+n)) \ge P(\bar{r}_m \le am, \bar{r}_{m+n} - \bar{r}_m \le an)$$
$$\ge P(\bar{r}_m \le am)P(\bar{r}_n \le an)$$

so $a_n = \log P(\bar{r}_m \le an)$ is superadditive. Repeating the first part of the proof of (7.1) shows everything except the fact that $\gamma > 0$.

To prove that $\gamma > 0$ we first prove the result for 1-dependent site percolation on $\mathscr{L}$. Let $C = \{y\text{:for some } m \le 0\ (m, 0) \to y\}$ and let $s_n = \sup\{m:(m, n) \in C\}$. The first step is to prove

(2) If $q < 1$ and $(1-p) < 3^{-36} \wedge 3^{-36/(1-q)}$ then

$$P(s_n \le nq) \le Ce^{-\gamma n}.$$

The proof of (2) involves another contour argument. We use the terminology and notation of Section 10. Let $D = \{(a, b) \in R^2: |a| + |b| \le 1\}$, let $W = \cup_{z\in C}(z + D)$ and let Γ be the boundary of the unbounded component of $(R \times (-1, n)) - W$ oriented in such a way that the segment from $(0, -1)$ to $(1, 0)$ is oriented in the direction indicated (see Figure 9).

Let m_1, m_2, m_3, m_4 be the number of segments in Γ of types 1, 2, 3, and 4 ($\nwarrow \swarrow \searrow \nearrow$). Since Γ starts as $(0, -1)$ and ends at $(s_n + 1, n)$, $m_3 + m_4 - m_1 - m_2 = s_n + 1$. The shortest possible contour has length $n + 1$. If Γ has length $n + 1 + k$ and $s_n \le qn$ we have

$$2(m_1 + m_2) = m_1 + m_2 + m_3 + m_4 - s_n - 1 \ge (1-q)n + k.$$

As in Section 10 we can conclude that there are at least $(m_1 + m_2)/2$ points associated with segments of Γ which must be closed and there is a subset of these sites of size $(m_1 + m_2)/18$ which are independent. Since there are at most 3^{n+k} contours with $n + k + 1$ segments we arrive at the estimate.

$$P(s_n \le qn) \le \textstyle\sum_{k=0}^{\infty} 3^{n+k}(1-p)^{((1-q)n+k)/36}$$
$$= 3^n(1-p)^{(1-q)n/36} \cdot (1 - 3(1-p)^{1/36})^{-1}$$

since $(1-p) < 3^{-36}$. The other restriction on p implies

$$3(1-p)^{(1-q)/36} < 1$$

and the proof of (2) is complete.

To deduce (1) from (2) we use the percolation construction from Section 5. It is at this point that we first need the ability to pick δ close to 0.

Proof of (1). If $a < \alpha$ we can pick $\delta < (\alpha - a)$ and $q < 1$ so that $q(\alpha - \delta) > a$ and then pick L so large that $P(\eta(z) = 1) > 1 - 3^{-36/(1-q+)}$. If $\bar{r}_n = \sup \xi_n^{(-\infty,0]}$ then for all $n \in [mL, (m+1)L)$

$$\bar{r}_n \geq s_m(\alpha - \delta)L - (1 + \delta)\alpha L.$$

It follows that for all m sufficiently large and all $n \in [mL, (m+1)L)$

$$P(\bar{r}_n \leq an) \leq P(s_m \leq qm) \leq Ce^{-\gamma m}.$$

From the proof above it follows easily that

(3) $$\alpha \quad \text{is continuous for} \quad p > p_c.$$

Proof of (3). In Section 3 we showed $\lim \sup_{p \to p_0} \alpha(p) \leq \alpha(p_0)$ so to prove (3) it suffices to show that if $a < \alpha(p_0)$ there is an $\varepsilon > 0$ so that if $p > p_0 - \varepsilon$, $\alpha(p) > a$. Pick δ, q, and L as in the proof of (2). Since the number of bonds in $R_{0,0}$ is finite it follows that we can pick $\varepsilon > 0$. If $p \in (p_0 - \varepsilon, p_0)$ then $P(\eta(z) = 1) > 1 - 3^{-36/(1-q^+)}$, and it follows from the computations above that $\alpha(p) \geq a$.

12. Bounds on $P(n \leq \tau^A < \infty)$ and $P(n \leq |C| < \infty)$. The first result we will prove in this section is

(1) If $p > p_c$ then there are constants C, γ so that

$$P(n < \tau^0 < \infty) \leq Ce^{-\gamma n}.$$

Proof. Applying (11.1) with $a = 0$ and summing from n to ∞ gives

$$P(\bar{r}_m \leq 0 \text{ for some } m \geq n) \leq C \exp(-\gamma_{11.1}(0)n)$$

so

$$P(\bar{\ell}_m < 0 < \bar{r}_m \text{ for all } m \geq n) \geq 1 - 2C \exp(-\gamma_{11.1}(0)n).$$

Now by (3.10d) we have $\tau_0 = \inf\{m \geq n : \bar{\ell}_m > \bar{r}_m\}$ on $\{\xi_n \neq \varnothing\}$ so we have proven the desired inequality with $\gamma = \gamma_{11.1}(0)$.

With (1) established it is easy to improve the result to

(2) If $p > p_c$ then there are constants C, γ (which are independent of A) so that

$$P(n < \tau^A < \infty) \leq C \exp(-\gamma n).$$

The proof is a "restart" argument. These arguments are based on two simple ideas.

(i) If you have a sequence of independent events with probability p then K, the number of failures before the first success, has a geometric distribution $P(K = n) = p(1-p)^n$, $n > 0$, and

(ii) If X_i is a sequence of independent random variables with $P(X_i \geq m) \leq C \exp(-\gamma m)$ and K is a random variable with a geometric distribution which has

$\{K = k\}$ independent of $(X_1, \cdots, X_k)$ then

$$P(X_1 + \cdots + X_K \geq m) \leq C' \exp(-\gamma m).$$

We will prove (ii) below.

To carry out the idea: Suppose A is finite (if A is infinite $P(\tau^A < \infty) = 0$). Let $x_0 = \sup A$ and let $N_1 = \inf\{n: \xi_n^{x_0} = \varnothing\}$. If $N_1 < \infty$ and $\xi_{N_1}^A = \varnothing$ let $x_1 = 0$. If $\xi_{N_1}^A \neq \varnothing$ let $x_1 = \sup \xi_{N_1}^A$. Let $\eta_n^1 = \{y: (x_1, N_1) \to (y, n)\}$ and let $N_2 = \inf\{n > N_1: \eta_n^1 = \varnothing\}$. By repeating the procedure above we can define x_k, η^k, and N_{k+1} if $k \leq K = \sup\{k: N_k < \infty\}$. On $\{\tau^A < \infty\}$, $N_K \geq \tau^A$ so $\{n \leq \tau^A < \infty\} \subset \{N_K \geq n\}$. To estimate $P\{N_K \geq n\}$ we observe (a) $P(K \geq k + 1 \mid K \geq k) = 1 - \rho$ so $P(K \geq k) = (1 - \rho)^k$, and (b) conditioned on $K > k$, $X_i = N_i - N_{i-1}$, $i = 1, 2, \cdots, k$ are independent with

$$P(X_i \geq n) = P(n \leq \tau^0 \mid \tau^0 < \infty) \leq C \exp(-\gamma n).$$

It is for this reason that we must ignore the death of ξ^A. From (b) it follows that we can pick $\theta > 0$ so that $\varphi(\theta) = E \exp(\theta X_i) < \infty$ and then $\varepsilon > 0$ so that $e^{-\theta}\varphi(\theta)^\varepsilon < 1$ to arrive at the estimate

$$P(X_1 + \cdots + X_{[\varepsilon n]} \geq n) \leq e^{-\theta n}\varphi(\theta)^{\varepsilon n}.$$

Combining this with the estimate from (a) gives

$$P(N_K \geq n) \leq P(K \geq \varepsilon n) + P(X_1 + \cdots + X_{\varepsilon n} \geq n) \leq C \exp(-\gamma n).$$

Since $|C_0| > n^2$ implies that $\tau^0 > n$ it follows immediately from (1) that

$$P(n < |C_0| < \infty) \leq C \exp(-\gamma n^{1/2}). \tag{3}$$

We proved a similar bound for $p < p_c$ in Section 7. This time the $n^{1/2}$ in the exponent is the right order of magnitude.

(4) If $p < 1$ there are constants $c, \Gamma \in (0, \infty)$ so that

$$P(n \leq |C_0| < \infty) \geq c \exp(-\Gamma n^{1/2}).$$

Proof. To prove this we need a result which we will prove in the next section

$$(1/n^2) \textstyle\sum_{m=1}^n (\xi_m^Z \cap [\bar{\ell}_m, \bar{r}_m]) \to \alpha\rho/2 \quad \text{a.s.}$$

(The proof of this result uses (1) above but not (4) so there is no circularity in the argument.) Let $\varepsilon < \rho/2$ and pick N large so that with probability $> 1 - \varepsilon$

$$\textstyle\sum_{m=1}^n (\xi_m^Z \cap [\bar{\ell}_m, \bar{r}_m]) > (\alpha\rho/3)n^2 \quad \text{for all} \quad n \geq N.$$

On $\xi_n \neq \varnothing$ we have

$$\textstyle\sum_{m=1}^n (\xi_m^Z \cap [\bar{\ell}_m, \bar{r}_m]) = \sum_{m=1}^n |\xi_m^0|$$

so

$$P(\textstyle\sum_{m=1}^n |\xi_m^0| > \alpha\rho n^2/3, \ \xi_n \neq \varnothing) \geq \rho/2.$$

(4) follows immediately since with probability $= (1 - p)^{2n+2}$, all the bonds up from the points $\{(x, n) \in \mathscr{L}: -n \leq x \leq n\}$ are closed.

REMARK. The simple argument given above is due to Aizenmann, Delyon, and Soulliard (1981) who proved a related inequality in the more difficult unoriented situation.

13. Correlation inequalities for ξ_∞^Z, limit laws for $|\xi_n^0|$. Let

$$\rho_n = P(\xi_n^0 \neq \varnothing) \quad \text{and} \quad \eta_n(x) = \xi_n^Z(x) - \rho_n.$$

In this section we will show

(1) If $p < p_c$ and $|x_1 - x_i| \geq 2m$ for $i = 2, \cdots, k$ then

$$|E \textstyle\prod_{i=1}^k \eta_n(x_i)| \leq 2P(m \leq \tau^0 < \infty).$$

This bound on the correlations will allow us to conclude that

(2) $$(1/n)|\xi_n^0| \to \alpha\rho \, 1_{\Omega_\infty} \quad \text{a.s.}$$

and make some progress toward a central limit theorem for $|\xi_n^0|$.

PROOF OF (1). The random variables $\xi_n^Z(x)$ have the same joint distributions as $1_{(\xi_n^x \neq \varnothing)}$ so it suffices to prove the bound for $\zeta(x) = 1_{(\xi_n^x \neq \varnothing)} - \rho_n$. If $n \leq m$ then $\zeta(x_1)$ is independent of $(\zeta(x_2), \cdots, \zeta(x_k))$ so

$$E \textstyle\prod_{i=1}^k \zeta(x_i) = E\zeta(x_1) E \prod_{i=2}^k \zeta(x_i) = 0.$$

If $m < n$ let

$$\zeta' = 1(\xi_m^{x_1} \neq \varnothing) - \rho_n$$

$$\zeta'' = 1(\xi_m^{x_1} \neq \varnothing) - \rho_m.$$

The random variables ζ'' and $(\zeta(x_2), \cdots \zeta(x_k))$ are independent so

$$E(\zeta'' \textstyle\prod_{i=2}^k \zeta(x_i)) = 0.$$

To prove the result all we have to do is estimate

$$|E \textstyle\prod_{i=1}^k \zeta(x_i) - E\zeta'' \prod_{i=2}^k \zeta(x_i)|.$$

To do this we proceed in two steps. First we observe

$$\begin{aligned} |\textstyle\prod_{i=1}^k \zeta(x_i) - \zeta' \prod_{i=2}^k \zeta(x_i)| &\leq |\zeta(x_i) - \zeta'| \, |\textstyle\prod_{i=2}^k \zeta(x_i)| \\ &\leq |\zeta(x_1) - \zeta'| = 0 \quad \text{or} \quad 1 \end{aligned}$$

and

$$P(\zeta(x_1) \neq \zeta') \leq |\rho_m - \rho_n|.$$

Therefore

$$|E \textstyle\prod_{i=1}^k \zeta(x_i) - E\zeta' \prod_{i=2}^k \zeta(x_i)| \leq |\rho_m - \rho_n|.$$

To estimate the other difference we observe

$$|\zeta' \textstyle\prod_{i=2}^k \zeta(x_i) - \zeta'' \prod_{i=2}^k \zeta(x_i)| \leq |\zeta' - \zeta''| = |\rho_m - \rho_n|.$$

Combining the two inequalities above proves the desired since in the case $m < n$

$$|\rho_m - \rho_n| = P(m \le \tau^0 < n).$$

Proof of (2). The first step is to observe that on $\Omega_\infty \xi_n^0 = \xi_n^Z \cap [\ell_n, r_n]$, $\ell_n/n \to -\alpha$, and $r_n/n \to \alpha$ so it is enough to show that

$$(1/n)\,|\,\xi_n^Z \cap [-\alpha n, \alpha n]\,| \to \alpha\rho \quad \text{as} \quad n \to \infty.$$

With a uniform bound on the correlations in ξ_n^Z this is easy to do. We let $S_n = \sum_{m=-\alpha n}^{\alpha n} \eta_n(x)$, $(\eta_m(x) = \xi_m^Z(x) - \rho_m)$ show that $ES_n^4 \le Cn^2$, and the conclusion follows from Chebyshev's inequality

$$(n\varepsilon)^4 P(|S_n| > n\varepsilon) \le Cn^2$$

and the Borel Cantelli lemma.

To bound ES_n^4 we observe

$$ES_n^4 = \textstyle\sum_{w,x,y,z} E\eta_n(w)\eta_n(x)\eta_n(y)\eta_n(z)$$

$$\le 24 \textstyle\sum_{w\le x\le y\le z} |E\eta_n(w)\eta_n(x)\eta_n(y)\eta_n(z)|$$

where the sum in both cases is over all possible points in $[-\alpha n, \alpha n]$. Now the number of terms in the sum with $\max\{|w-x|, |y-z|\} = 2m$ is $\le 2(m+1)n^2$. (The number of ways we can pick w, z is $\le n^2$ since $\alpha \le 1$ and only every other point is possible. Given the choice of w and z there are at most $2(m+1)$ ways of picking x and y.) Combining the last observation with (1) and the bound $P(m \le \tau^0 < \infty) \le C\exp(-\gamma m)$ proved in the last section shows

$$ES_n^4 \le 96 \left(\textstyle\sum_{m=0}^{\infty} (m+1)C\exp(-\gamma m)\right)n^2 = Cn^2$$

and completes the proof of (2).

By using (2) we can prove the result we used in the last section

(3) $$(1/n^2) \textstyle\sum_{m=1}^{n} (\xi_m^Z \cap [\bar{\ell}_m, \bar{r}_m]) \to \alpha\rho/2 \quad \text{a.s.}$$

Proof. As above it suffices to show

$$(1/n^2) \textstyle\sum_{m=1}^{n} (\xi_m^Z \cap [-\alpha m, \alpha m]) \to \alpha\rho/2 \quad \text{a.s.}$$

To do this let $X_m = (1/m)\,|\,\xi_m^Z \cap [-\alpha m, \alpha m]\,|$ and observe that $X_m \to \alpha\rho$ a.s. and

$$\frac{1}{n^2} \textstyle\sum_{m=1}^{n} (\xi_m^Z \cap [-\alpha m, \alpha m]) = \frac{1}{n} \sum_{m=1}^{n} X_m \frac{m}{n} \to \alpha\rho \int_0^1 x\,dx = \frac{\alpha\rho}{2}.$$

With the strong law for ξ_n^0 established, it is natural to think about central limit theorems, especially since the random variables have positive correlations in the sense of (5.6). We can apply results of Newman (1980) and Newman and Wright (1981) to conclude

(4) $$(1/\sqrt{n})(|\,\xi_n^Z \cap [-\alpha n, \alpha n]\,| - \alpha\rho n) \Rightarrow N(0, \sigma^2)$$

where

$$\sigma^2 = \textstyle\sum_x \operatorname{cov}(\xi_\infty^Z(0), \xi_\infty^Z(x)) < \infty.$$

PROOF. This is almost a consequence of Theorem 3 of Newman and Wright (1981). Their result concerns a single sequence of random variables while (4) requires a result for triangular arrays. The reader can see from the fact that the bound in Theorem 1 is stated in terms of covariances that the details of their proof generalize to triangular array setting so we will not repeat the details here.

In the proof of (2) a strong law for $\xi_n^Z \cap [-\alpha n, \alpha n]$ was good enough to get a strong law for ξ_n^0. Unfortunately here a central limit theorem for the first quantity is not good enough to get one for the second because

$$(5) \qquad |\xi_n^Z \cap [\bar{\ell}_n, \bar{r}_n]| - |\xi_n^Z \cap [-\alpha n, \alpha n]| \approx \rho \frac{-\alpha n - \bar{\ell}_n}{2} + \rho \frac{\bar{r}_n - \alpha n}{2}$$

so if, as we suspect,

$$(6) \qquad \frac{\bar{r}_n - \alpha n}{\sqrt{n}} \Rightarrow N(0, \tau^2)$$

the differences in (5) will contribute to the limit.

14. Infinite differentiability of $P(\Omega_\infty)$ for $p > p_c$. In this section we will show that the probability of percolation is infinitely differentiable for $p > p_c$. The first step in doing this is to write a series for the probability of no percolation

$$(1) \qquad P(\Omega_\infty^c) = \textstyle\sum_{n,m} a_{n,m} p^n (1-p)^m$$

where $a_{n,m}$ is the number of clusters with n open bonds and a boundary which consists of m closed bonds. The meaning of boundary should be clear from (1), but if you want a precise definition, a bond $x \to y$ is in the boundary if it is closed and x can be reached from 0. To illustrate the definition of boundary we have drawn some of the small clusters in Figure 10. The solid lines are the open bonds which make up the cluster, and the dotted lines are the closed bonds which make up the boundary.

Given the expression in (1) our strategy for proving $P(\Omega_\infty)$ is differentiable is clear. We need to show the derivative of the sum is the sum of the derivatives. The first step is to differentiate the individual terms

$$\frac{d}{dp} p^n (1-p)^m = np^{n-1}(1-p)^m - mp^n(1-p)^{m-1} = \left(\frac{n}{p} - \frac{m}{1-p}\right) p^n (1-p)^m$$

and then to check that the expression for the derivative converges absolutely

$$\textstyle\sum_{n,m} a_{n,m} \left| \frac{n}{p} - \frac{m}{1-p} \right| p^n(1-p)^m \le E\left(\frac{|C_0|}{p} + \frac{|\partial C_0|}{1-p}; |C_0| < \infty\right) < \infty$$

and uniformly in compact subsets of $(p_c, 1)$ since

$$P(\infty > |C_0| + |\partial C_0| > (n+1)^2) \le P(n < \tau^0 < \infty).$$

The last computation is not far from a proof that $P(\Omega_\infty)$ is differentiable for

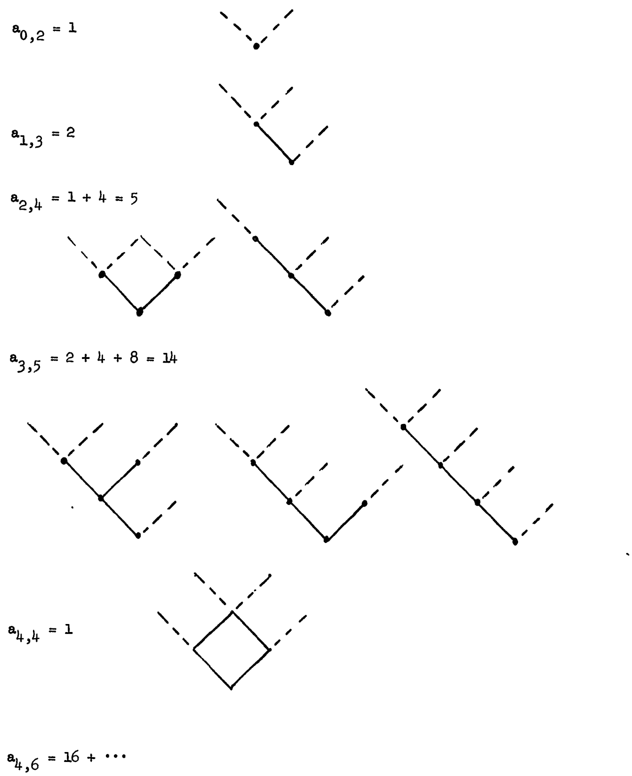

FIG. 10.

$p \in (p_c, 1)$. Let $f_{n,m}(p) = a_{n,m}p^n(1-p)^m$ and observe that for fixed, n, m,

(a) $f'_{n,m}(p)$ is continuous and

$$f_{n,m}(p+h) - f_{n,m}(p) = \int_p^{p+h} f'_{n,m}(x)\,dx$$

and

(b) if $\varepsilon > 0$ and δ is chosen so that $[p-\delta, p+\delta] \subset (p_c, 1)$ we can pick N so large that for all $x \in [p-\delta, p+\delta]$

$$\textstyle\sum_{n \ge N; m} |f'_{n,m}(x)| \le \varepsilon.$$

With these two observations in mind it is routine to establish that as $h \to 0$

$$1/h\ \left(\textstyle\sum_{n,m} f_{n,m}(p+h) - \sum_{n,m} f_{n,m}(p) - \sum_{n,m} f'_{n,m}(p)\right) \to 0$$

and the details are left to the reader.

To show that the second derivative exists we let

$$f_{n,m}(p) = \left(\frac{n}{p} - \frac{m}{1-p}\right)p^n(1-p)^m$$

and observe that

$$f'_{n,m}(p) = \left(\frac{n}{p} - \frac{m}{1-p}\right)^2 p^n(1-p)^m - \left(\frac{n}{p^2} + \frac{m}{(1-p)^2}\right)p^n(1-p)^m$$

so (a) and (b) are still satisfied, and we can repeat the proof above to show that $P(\Omega_\infty)$ is twice differentiable. The same proof works for higher derivatives and it follows that $P(\Omega_\infty)$ is infinitely differentiable for $p \in (p_c, 1)$.

To get infinite differentiability at $p = 1$ we have to find a way of bounding $f'_{m,n}(p)$ in a neighborhood of 1. To do this we observe

$$\textstyle\sum_n a_{n,m} = \text{no. of clusters with boundary of size } m.$$

By observations in Section 10 if the boundary has size m the associated contour has length $\le 2m$ (at least half the bonds cut by the contour must be part of the boundary) so

$$\textstyle\sum_n a_{n,m} \le \sum_{k=2}^{m} 3^{2k} \le C\, 3^{2m}.$$

Writing the derivative as

$$\textstyle\sum_{m,n} a_{n,m}(np^{n-1}(1-p)^m - mp^n(1-p)^{m-1})$$

and using the estimate above we see that if $p > 8/9$

$$\textstyle\sum_{m,n} a_{n,m} mp^n(1-p)^{m-1} \le \sum_m C\, 3^{2m}(1-p)^{m-1} < \infty$$

and since $n \le m^2$

$$\textstyle\sum_{m,n} a_{n,m} np^{n-1}(1-p)^m \le \sum_m C\, 3^{2m} m^2(1-p)^m < \infty.$$

The last two estimates show that we can pick M so large that for all $x \in [.9, 1]$

$$\textstyle\sum_{m>M;n} |f'_{n,m}(x)| \ge \varepsilon$$

so again (a) and (b) are satisfied and we can conclude $P(\Omega_\infty)$ is differentiable at 1, etc.

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