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`rough notes`

Abstract

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1 Introduction

xxx the model is "made up" to be mathematically interesting. Use "cities" as convenient language, not literally.

The model At each time $t = 1, 2, 3, \ldots$ there are cities at positions x_i in the unit square $[0,1]^2$, with populations $N_i(t) \geq 1$, the total population being $\sum_i N_i(t) = t$. The model has three parameters

$$0 < c_0 < \infty, \quad 0 < \alpha < \infty, \quad \beta > 0$$

(though we focus on the case $0 < \alpha < 1$) which are used to define a function

$$I_0(n,r) = c_0 n^\alpha r^{-\beta} \tag{1}$$

interpreted as the "influence" of a city of population n at distance r from the city. For a position $y \in [0,1]^2$ define

$$I(y,t) = \max_i I_0(N_i(t), |y - x_i|) = c_0 \max_i N_i^\alpha(t) |y - x_i|^{-\beta} \tag{2}$$

and then define the *sphere of influence* of city i to be

$$\mathcal{S}(i,t) = \{y : I_0(N_i(t), |y - x_i|) = I(y,t)\} \tag{3}$$

where the convention about boundaries between cities' spheres is unimportant. At time 1 there is a single city of population 1 at a uniform random point of $[0,1]^2$. The general evolution rule is:

At time $t+1$ an immigrant arrives at a uniform random position U in $[0,1]^2$, and either
(i) (with probability $1/(1 + I(U,t))$) founds a new city at position U with population 1;
or (ii) (with probability $I(U,t)/(1 + I(U,t))$) joins the city i for which $U \in \mathcal{S}(i,t)$, thereby increasing its population to $N_i(t+1) = N_i(t) + 1$.

Remarks If city populations were equal then the partition into spheres of influence is just the usual Voronoi tessellation [10]; so in general it is a form of weighted Voronoi tessellation.

Comments on model The two qualitative features of the model are
the growth rate of a city depends on its size and on the sizes and distances of other cities
some stochastic rule for founding of new cities.
One could imagine any different rules to formalize these features; while there is no necessary connection between the two features, we are using a "slick" formulation in which both are xxx via the same influence function.

Heuristic analysis Take $0 < \alpha < 1$. We study the order of magnitude of several quantities. Suppose there are $M(t)$ cities at time t, and suppose their populations are mostly the same order of magnitude. Then typical city population $N^*(t) \approx t/M(t)$ and distance $R^*(t)$ from typical point to nearest city is $\approx M^{-1/2}(t)$, so the typical value $I^*(t)$ of the influence function $I(y,t)$ is $\approx (\frac{t}{M(t)})^\alpha \ M^{\beta/2}(t) = t^\alpha M^{-\alpha+\beta/2}(t)$. The chance that a new arrival founds a new city is about $1/I^*(t)$, so we get an equation

$$\tfrac{dM}{dt} \approx \tfrac{1}{I^*(t)} \approx t^{-\alpha} M^{-\beta/2+\alpha}. \tag{4}$$

This has solution

$$M(t) \approx t^\theta, \quad \theta = \frac{1-\alpha}{1-\alpha+\beta/2}$$

obtained from solving $\theta - 1 = -\alpha + \theta(\alpha - \beta/2)$. Note that the typical influence is therefore

$$I^*(t) \approx (dM(t)/dt)^{-1} \approx t^{1-\theta}; \quad 1-\theta = \tfrac{\beta}{2-2\alpha+\beta} \tag{5}$$

and the typical distance to nearest city is

$$R^*(t) \approx M^{-1/2}(t) \approx t^{-\theta/2}; \quad \theta/2 = \tfrac{1-\alpha}{2-2\alpha+\beta} \tag{6}$$

and the typical city population size is

$$N^*(t) \approx \tfrac{t}{M(t)} \approx t^{1-\theta}; \quad 1-\theta = \tfrac{\beta}{2-2\alpha+\beta}. \tag{7}$$

The heuristic argument above rests upon an intuitive picture of the qualitative behavior of the process, that for large t and a typical position y
(a) most different cities' populations are the same order of magnitude
(b) y is in the sphere of influence of some *nearby* city
(c) a city newly founded at t will grow, in time δt, to some population which is $\varepsilon(\delta)$ times the typical time-t city population.

Call this the *balanced growth scenario*. But one can imagine an alternative picture, the *unbalanced growth scenario*, in which
(d) y is in the sphere of influence of some city A at distance r which is much larger than the distance to nearby cities
(e) the nearby cities' populations are a smaller order of magnitude than city A's, and their spheres of influence are surrounded by that of city A.

To investigate these scenarios we try a self-consistency check. Consider a city founded at time t, and consider $N(s)$ = population of this city at time s after founding, looked at over a relatively short time period $0 < s < \frac{1}{100}t$, say. The radius $r(s)$ of its sphere of influence satisfies

$$N^\alpha(s) r^{-\beta}(s) \approx I^*(t) \approx t^{1-\theta}.$$

The rate of population growth is proportional to area of sphere of influence, so

$$\frac{dN(s)}{ds} \approx r^2(s) \approx t^{-2(1-\theta)/\beta} \ N^{2\alpha/\beta}(s); \quad N(0) = 1.$$

We now have two cases.

Case 1. $\beta < 2\alpha$.
The solution of $dy(s)/ds = y^{2\alpha/\beta}(s)$ explodes in finite time s, but stays bounded for some small time, and so $N(s)$ stays bounded for some time s of order $t^{2(1-\theta)/\beta}$. But the assumption $\beta < 2\alpha$ implies $2(1-\theta)/\beta = \frac{1}{1-\alpha+\beta/2} > 1$ implying that $N(\frac{1}{100}t)$ is bounded, in contradiction to behavior (c) above.

Case 2. $\beta > 2\alpha$.
Here the solution is

$$N(s) \approx t^\xi (s + t^{-\xi/\phi})^\phi; \quad \phi = \frac{\beta}{\beta - 2\alpha}, \quad \xi = \frac{2(\theta-1)}{\beta - 2\alpha}.$$

Here $-\xi/\phi$ works out to be $\frac{1}{1-\alpha+\beta/2} < 1$ and so $N(\frac{1}{100}t)$ is order $t^{\xi+\phi}$. Then (somewhat magically?) a calculation shows $\xi + \phi = 1 - \theta$, consistent with behavior (c) above.

So the conclusion of these heuristics is that we predict (recall $0 < \alpha < 1$)

> the balanced growth scenario holds when $\beta > 2\alpha$
> the unbalanced growth scenario holds when $\beta < 2\alpha$.

One can *a posteriori* see a conceptually simpler distinction between the two cases. Consider a city founded at time t. If the area of its sphere of influence upon founding is $> 1/t$ then it will tend to grow faster (proportional to size) than average, while if this area is smaller than $1/t$ it will tend to grow slower. But to calculate this area one needs some information about the behavior of the process, so we do not see any simple *a priori* heuristic that these two cases are $\beta < 2\alpha$ and $\beta > 2\alpha$.

In the unbalanced case we envisage that most of the population is in one city, or in a small number of cities, and so the analogs of (5 - 7) are

$$N^*(t) = t^{1-o(1)}; \quad R^*(t) = t^{-o(1)}; \quad I^*(t) = t^{1-o(1)}.$$

Note that these exponents are therefore discontinuous as (α, β) cross the boundary between the balanced and unbalanced regions.

In the case $\alpha > 1$ we expect unbalanced growth (the solution of $dy(s)/ds = y^{2\alpha/\beta}(s)$ explodes in finite time s). The case $\alpha = 1$ can be considered *critical*, in that growth rate is proportional to population size. This case is loosely analogous to other models (see Gibrat's law (section 1.2) and the Chinese restaurant processs (section 1.2).

To summarize: heuristics suggest the parameter space should divide into three regions:

$0 < \alpha < 1$ and $\beta > 2\alpha$	balanced growth
$\alpha > 1$ or $\beta < 2\alpha$	unbalanced growth
$\alpha = 1$ and $\beta > 2\alpha$	critical growth

and this guides our theoretical development.

1.1 Methodology

Our methodology is to seek statistics $S(t)$ of the process for which we can control (upper or lower bound) the incremental conditional expectation $E(S(t+1) - S(t)|\mathcal{F}(t))$ and thereby upper or lower bound the growth rate of $S(t)$ by martingale methods. Conceptually we imagine $I(y,t)$ as a time-varying "random environment" and study how the environment affects growth of city population and distance to nearest city, and how these changes feed back into changing the environment. The fact that $I(y,t)$ is monotone in t is technically helpful.

xxx in balanced case, end up with a bunch of inequalities between rigorously-defined upper and lower exponents for growth rates. In other cases, scattered results.

xxx state results!

As described in section 6.4 there is a certain associated dynamical system whose study might lead to stronger results. xxx future work!

1.1.1 Methodological difficulties

xxx geometry. Regardless of the parameter values, the first $1,000$ (say) immigrants might go to one city, or $1,000$ different cities, or 50 cities in one corner of the square, ... xxx hard to track evolution from atypical start.

Here are some specific intuitive ideas which we have not been able to implement.

Coupling When the time-$t+1$ immigrant arrives at a given position, there are two alternatives (join existing city or found new city). One might hope that the two conditioned future processes could be coupled in some manageable way (e.g. the second alternative differs from the first only by transferring some inhabitants of other cities to the new city). But we do not have any rigorous coupling construction of this type.

$0-1$ **laws** We do not know any general result saying that the asymptotic behavior of the process is unaffected by initial behavior; and indeed in certain settings we know $0-1$ laws do not hold (see remarks after Theorem 2).

Repulsion of cities Because the influence function is infinite at a city, the positions of cities should intuitively be "less clustered" than random cities. But we don't know any simple formalization of this idea.

1.2 Related work

We do not know any closely related previous work, but here are some more distantly related topics.

Random processes with reinforcement See [11] for an elegant survey of such processes, generalizing classical urn models. Such models typically do not have spatial structure, or else spatial structure is represented by edges of a graph.

The Chinese restaurant process In our terminology, this is the process where the arrival at time $t+1$ either
(i) (with probability $\theta/(t+\theta)$) founds a new city with population 1;
or (ii) (with probability $N_i(t)/(t+\theta)$) joins city i.
See [12] for a treatment of this model and generalizations; these do not involve any spatial structure. A key feature of this model is that there exists a limit for normalized ordered city sizes

$$t^{-1}(N_{(1)}(t), N_{(2)}(t), \ldots) \stackrel{d}{\to} (X_1, X_2, \ldots), \quad X_i > 0, \ \sum_i X_i = 1 \tag{8}$$

where the limit has Poisson-Dirichlet distribution. The $\alpha = 1$ case of our model is somewhat analogous, so it is natural to ask whether it has the same behavior (8).

xxx add discussion. In the balanced ($\beta > 2$) case the growth seems to be slightly sublinear (Theorem 23); could conjecture (8) holds for $\beta < 2$ but I haven't thought about it yet ...

Note we could consider a variant of our model in which an immigrant at position y may join any city i, with probabilities proportional to $I_0(N_i(t), |y - x_i|)$; we have not studied this variant, which is more similar to the Chinese restaurant process.

Random Johnson-Mehl tessellations [9] . In this model (in our language) cities appear as a space-time Poisson process (x_i, t_i), and we imagine a wave traveling at speed one in all directions starting from x_i at time t_i; the cell of the tessellation of x_i is the set of points which are reached by the wave from x_i before being reached by the wave from any $x_j, \ j \neq i$. So (indentifying cell areas A_i with population) this gives a "static" model of positions and populations (x_i, A_i) of cities which involves spatial interaction.

Coagulation models There is a large literature in physical chemistry on *coagulation*, meaning coalescence of clusters of mass. Though the underlying picture is of motion in space (with coalescence when clusters meet), the usual models [1] ignore spatial position and study deterministic equations for the density $f_i(t)$ of mass-i clusters at time t; a parameter in the equations is a kernel $K(i, j)$ giving the propensity for mass-i and mass-j clusters to merge. Closest to our model is the special case of the Becker-Döring equations [8] of polymers growing by collisions with monomers; mass -i clusters can grow only by coalescing with mass-1 clusters.

A spatial network model The only model we have found fitting our introductory xxx is one studied by simulation in [2]. This model has additional graph structure, but (interpreting their "number of edges" as "population") is essentially the following model. Take an integer parameter $m \geq 1$ and a "distance scale" parameter r_c.

(i) A city arrives at a uniform random point y in a given region, and is given population m.
(ii) Simultaneously m existing cities have their population increased by 1, with city i chosen with probability proportional to $N_i \exp(-|y - x_i|/r_c)$.

This model was intended as a spatial analog of the "proportional attachment" models popular in the statistical physics of complex networks literature. Their conclusions focus on network traffic properties, and so are not comparable to ours.

Real-world cities: size and spatial distribution Two classic notions from economic geography are

Zipf's law: that the distribution of city size N follows (for larger cities) approximately the power law $P(N > n) \approx cn^{-1}$ of exponent 1;

Gibrat's law: that the (percentage) growth rate of cities is independent of their size.

See [5] for discussion. Empirical studies [6] generally find Zipf's law to be quite accurate and Gibrat's law to be more crudely realistic. The simplest stochastic model for Gibrat's law is that percentage growth in successive time units is i.i.d. (over cities and times). This model of course leads to a log-normal distribution for city size; attempts to reconcile mathematically the two laws have led to a body of literature [5] which the authors find rather confused.

A modern big picture of spatial aspects of economics is provided by the textbook [4]. We do not know any treatment in economic geography of an explicitly stochastic model of cities sizes and positions. Interesting empirical study in [7] concludes

> City location is essentially a random process and interactions between cities do not help determine the size of a city. Both of these findings contradict our theoretical priors about the role of geography (physical and economic) in determining city outcomes.

1.3 Notation

Here we fix notation for the rest of the paper. Index cities as $i = 1, 2, \ldots$ in their order of founding. The state of the process at time t can be described as $(N_i(t), x_i; 1 \le i \le M(t))$ where $M(t)$ is the number of cities at t, and $N_i(t)$ and x_i denote the population and position of city i. Write the *ordered* city sizes as $N_{(1)}(t) \ge N_{(2)}(t) \ge \ldots \ge N_{M(t)}(t) \ge 1$. Write $(\mathcal{F}(t))$ for the natural filtration. Recall from (2,3) the *influence* random field

$$I(y,t) = c_0 \max_i N_i^\alpha(t)|y - x_i|^{-\beta}$$

and the *sphere of influence* of city i

$$\mathcal{S}(i,t) = \{y : N_i^\alpha(t)|y - x_i|^{-\beta} = \max_j N_j^\alpha(t)|y - x_j|^{-\beta}\}.$$

The definition of process dynamics says

$$\begin{aligned} P(N_i(t+1) = N_i(t) + 1|\mathcal{F}(t)) &= \int_{\mathcal{S}(i,t)} \frac{I(y,t)}{1 + I(y,t)}\, dy \\ P(M(t+1) = M(t) + 1, N_{M(t+1)}(t+1) = 1, x_{M(t+1)} \in dy|\mathcal{F}(t)) &= \frac{1_{(y \in \mathcal{S}(i,t))}}{1 + I(y,t)}\, dy \\ \sum_{i=1}^{M(t)} N_i(t) &= t. \end{aligned}$$

We also study

$$R(y,t) = \text{ distance from } y \text{ to closest city at time } t.$$

We write $D(y,r)$ for the disc of center y and radius r.

Convention: Write $\kappa < \infty$ and $\kappa^{-1} > 0$ for constants depending only on the model parameters (c_0, α, β).

2 The unbalanced scenario

In the case $\alpha > 1$ or $\beta < 2\alpha$ we heuristically expect most of the population to be attracted into one or several large cities. There are many possible ways to xxx; here is our conjecture for xxx. Note that the number $M(t)$ of cities, and the population $t - N_{(1)}(t)$ outside the largest city, are increasing in t, so there exist a.s. limits

$$M(\infty) := \lim_{t\to\infty} M(t) \le \infty; \quad N_{(\ge 2)}(\infty) := \lim_{t\to\infty} (t - N_{(1)}(t)) \le \infty.$$

Conjecture 1 *(a) If $\alpha > 1$ then $M(\infty) < \infty$ a.s.*
(b) If $\beta < 2\alpha$ then $N_{(1)}(t)/t \to 1$ a.s. as $t \to \infty$.
(c) If $\alpha > 1$ and $\beta < 2\alpha$ then $N_{(\ge 2)}(\infty) < \infty$ a.s.

What we can actually prove is somewhat weaker.

Theorem 2 *(a) If $\alpha > 1$ then $P(N_{(\ge 2)}(\infty) = 0) > 0$.*
(b) If $\beta < 2\alpha$, then for all $\varepsilon > 0$

$$P(N_{(1)}(t) > t^{1-\varepsilon}) \to 1 \text{ as } t \to \infty.$$

(c) Suppose $\alpha < 1$ and $\beta < 2\alpha$. Then there exists κ such that

$$P(N_{(\ge 2)}(t) \le \kappa t^{1-\alpha} \ \forall t) > 0$$

and hence

$$P(N_{(1)}(t)/t \to 1) > 0.$$

Remarks. (i) Of course it is possible that the first $t+1$ arrivals form different cities, so $P(N_{(\ge 2)}(\infty) \ge t) \ge P(M(\infty) \ge t+1) > 0$ for each $t < \infty$. Thus Theorem 2(a) shows there can be no simple $0-1$ law for tail events, and we see no simple way to improve the conclusion of (c) to "with probability one".

(ii) It is easy to see that

$$M(\infty) < \infty \text{ a.s. if and only if } E\text{area } \{y : \sum_t \frac{1}{I(y,t)} < \infty\} = 1.$$

So by considering the influence of the largest city, we see

$$\text{if } \alpha > 1 \text{ and if } \lim_t \tfrac{N_{(1)}(t)}{t^{1-\varepsilon}} = \infty \text{ a.s. for all } \varepsilon > 0 \text{ then } M(\infty) < \infty \text{ a.s.} \tag{9}$$

In the case $\beta < 2\alpha$, Theorem 2(b) shows $\lim_t \frac{N_{(1)}(t)}{t^{1-\varepsilon}} = \infty$ *in probability*; can we improve this to "a.s."?

(iii) To study the conjectures in the case $\beta < 2\alpha$, the conjectured instability of the associated dynamic system (Conjecture 32) may be relevant.

Proof of Theorem 2(a) This is easy, by considering the event "at time t the whole population is in one city" and observing
(i) $P(N_1(t) = t) > 0$ for each $t \geq 1$.
(ii) $P(N_1(t+1) = t+1 | \mathcal{F}_t) \geq 1 - \kappa t^{-\alpha}$ on $\{N_1(t) = t\}$
for $\kappa = 2^{\beta/2}/c_0$. These imply

$$\text{if } \alpha > 1 \text{ then } P(N_1(t) = t \ \forall t) > 0$$

as required. ∎

2.1 Proof of Theorem 2(c)

We start with a simple fixed-time geometry lemma. Let N_A and x_A be the population and position of city A.

Lemma 3 *(a) If $N_A > N_B$ then $\{y : I(B, y) \geq I(A, y)\}$ is contained in the disc with center x_B and radius r solving*

$$1 + \tfrac{|x_A - x_B|}{r} = (N_A/N_B)^{\alpha/\beta}. \tag{10}$$

(b) If $N_A/N_B \geq \kappa := (1 - \pi^{1/2}/4)^{-\beta/\alpha}$ then the area of the sphere of influence of B is at most $8(N_B/N_A)^{2\alpha/\beta}$.

Proof. For any point y with $|y - x_B| = s$,

$$\frac{I(B, y)}{I(A, y)} \leq \left(\frac{N_B}{N_A}\right)^{\alpha} \left(\frac{s + |x_A - x_B|}{s}\right)^{\beta}.$$

For the left side to be ≥ 1, we require s to be smaller than the solution r of

$$1 = \left(\frac{N_B}{N_A}\right)^{\alpha} \left(\frac{r + |x_A - x_B|}{r}\right)^{\beta}$$

which reduces to (10). For (b), the choice of κ is such that $N_A/N_B \geq \kappa$ implies $(N_A/N_B)^{\alpha/\beta} - 1 \geq \sqrt{\pi/4}\ (N_A/N_B)^{\alpha/\beta}$. Then the solution r of (10) satisfies

$$r \leq |x_A - x_B|\ \sqrt{4/\pi}(N_B/N_A)^{\alpha/\beta} \leq \sqrt{8/\pi}(N_B/N_A)^{\alpha/\beta}$$

and the area in question is at most πr^2. ∎

The next lemma studies a process which will be used for comparison.

Lemma 4 *Let $\alpha > 0$, $\gamma > 1$. Consider the continuous time particle process on sites $\{1, 2, 3, \ldots\}$, starting at time t_0 with no particles, where particles arrive at site 1 at rate $t^{-\alpha}$, and where a particle at site j jumps to site $j+1$ at rate $(j/t)^{\gamma}$. Suppose j is sufficiently large that $\alpha + (\gamma - 1)(j-1) > 1$. Then*

$$E(\text{ number of particles which ever reach site } j\ ) \to 0 \text{ as } t_0 \to \infty.$$

Proof. First consider the continuous-time Markov chain on sites $\{1, 2, 3, \ldots\}$, with transitions $j \to j+1$ at rate j^γ, and started at time 0 at site 1. Let T_j be the first hitting time to site j. Because T_j is the sum of $j-1$ independent Exponentials,

$$P(T_j \le u) = O(u^{j-1}) \text{ as } u \downarrow 0.$$

Now consider a particle in the particle process arriving at time $t_1 > t_0$, and write $h_j(t_1)$ for the chance it ever reaches site j. Its motion is a deterministic time-change of the Markov chain, with time $\tau > t_1$ for the particle process corresponding to time

$$\int_{t_1}^{\tau} t^{-\gamma}\, dt = \frac{t_1^{1-\gamma} - \tau^{1-\gamma}}{\gamma - 1}$$

for the Markov chain. So

$$h_j(t_1) = P\left(T_j < \tfrac{t_1^{1-\gamma}}{\gamma-1}\right) = O\left(t_1^{(1-\gamma)(j-1)}\right) \text{ as } t_1 \to \infty.$$

Now

$$E(\text{ number of particles which ever reach site } j\ ) = \int_{t_0}^{\infty} t^{-\alpha} h_j(t)\, dt$$

and the result follows. ∎

Returning to our "cities" process, set $\gamma = 2\alpha/\beta > 1$. Fix large t_0 and suppose the first t_0 arrivals join the same city. Let $C(t)$ be the population of this city. Fix large K and consider the event

$$B_t := \{s - C(s) \le K s^{1-\alpha}, 1 \le s \le t\}.$$

Given K, by choosing t_0 sufficiently large we can assume

$$B_t \subseteq \{C(s) \ge s/2, 1 \le s \le t\}. \tag{11}$$

xxx rest of argument only sketched. key points are that we are working on B_t at each t, and that constants κ and j_0 do not depend on K.

On the event B_t, using (11),

$$\min_y I(y,t) \ge \kappa^{-1} t^\alpha \ \forall t. \tag{12}$$

We now claim that the process of "populations of other cities" evolves as a discrete-time analog of the particle process in Lemma 4. That is, at time t

(i) chance $\le \kappa t^{-\alpha}$ to found a new city
(ii) for a city of population j, chance $\le \kappa (j/t)^\gamma$ to increase to $j+1$

the former by (12) and the latter by Lemma 3(b). So by the conclusion of Lemma 4 (xxx modified to discrete time and the extra constants) we can choose j_0 and t_0 such that the probability that any other city ever reaches population j_0 is less than $1/100$. The expected number of new cities up to time t is by (i) bounded by $\kappa t^{1-\alpha}$ and so (outside the event of probability $< 1/100$) the expected total population of the other cities is bounded by $\kappa j_0 t^{1-\alpha}$. Now specify

$$K = 4\kappa j_0$$

and we have shown

$$P(B_t) \ge \tfrac{99}{100} - \tfrac{1}{4}.$$

This gives the result, recalling (12).

2.2 Proof of Theorem 2(b)

Note that Lemmas 5 - 7 are true for all parameter values.

Lemma 5

$$\min_y I(y,t) \uparrow \infty \ \textit{a.s. as} \ t \to \infty.$$

Proof. Fix K and define $r(K)$ by

$$c_0(3r(K))^{-\beta} = K.$$

Fix a position y. We must have

$$\sup_{u \in D(y,r(K))} I(u,t) \uparrow \infty \text{ a.s. as } t \to \infty$$

otherwise cities would be founded in the disc, making the *sup* be infinite. If the nearest city to y is at distance $\geq 2r(K)$ then the ratio $\frac{\sup_{u \in D(y,r(K))} I(u,t)}{\inf_{u \in D(y,r(K))} I(u,t)}$ is bounded, and hence

$$P(R(y,t) \geq 2r(K), \inf_{u \in D(y,r(K))} I(u,t) \leq K) \to 0.$$

But by definition of $r(K)$

$$\text{if } R(y,t) < 2r(K) \text{ then } \inf_{u \in D(y,r(K))} I(u,t) > K$$

and so we have shown that for each y

$$P(\inf_{u \in D(y,r(K))} I(u,t) \leq K) \to 0.$$

The result now follows by compactness (apply to a finite $r(K)$-dense set (y_j)). ■

Lemma 6 *For each* $K < \infty$

$$E \sum_i a_i(t) 1_{(N_i(t) \leq K)} \to 0.$$

Proof. Fix K and define

$$W_t = \sum_i a_i(t)(K + 1 - N_i(t))^+.$$

We want to upper bound the increment $W_{t+1} - W_t$. There are three cases for the arrival at time $t+1$.
(i) If the arrival joins a city with population $> K$ then $W_{t+1} - W_t \leq 0$.
(ii) If the arrival joins a city i with population $1 \leq N_i(t) \leq K$ then $W_{t+1} - W_t \leq -a_i(t)$.
(iii) If the arrival founds a new city whose sphere of influence has area a^* then $W_{t+1} - W_t \leq K a^*$.
Write $I_* = \min_y I(y,t)$. In case (iii) the sphere of influence is contained within a disc of radius r satisfying $c_0 r^{-\beta} = I_*$ and so $a^* \leq \pi(c_0/I_*)^{2/\beta}$. Thus

$$\begin{aligned} E(W_{t+1} - W_t | \mathcal{F}_t) &\leq -\tfrac{I_*}{I_*+1} \sum_i a_i(t) 1_{(N_i(t) \leq K)} + K\pi(c_0/I_*)^{2/\beta} \\ &\leq -\tfrac{I_*}{I_*+1} \tfrac{W_t}{K} + K\pi(c_0/I_*)^{2/\beta}. \end{aligned}$$

Fix t_0 and J, and for $t \geq t_0$ consider

$$w(t) = EW_t 1_{(\min_y I(y,t_0) \geq J)}.$$

Then

$$w(t+1) - w(t) \leq -\tfrac{J}{J+1} \tfrac{w(t)}{K} + K\pi(c_0/J)^{2/\beta}$$

which implies

$$\limsup_t w(t) \leq \tfrac{J+1}{J} K^2 \pi (c_0/J)^{2/\beta}.$$

Observe

$$\sum_i a_i(t) 1_{(N_i(t) \leq K)} \leq W_t 1_{(\min_y I(y,t_0) \geq J)} + 1_{(\min_y I(y,t_0) < J)}.$$

So

$$\limsup_t E \sum_i a_i(t) 1_{(N_i(t) \leq K)} \leq \tfrac{J+1}{J} K^2 \pi (c_0/J)^{2/\beta} + P(\min_y I(y,t_0) < J).$$

By Lemma 5 we can let $t_0 \to \infty$ to make the final term $\to 0$, and then letting $J \to \infty$ we see the right side equals 0. ■

Lemma 7 *$E \int \log I(y,t)\ dy \geq (\alpha - o(1)) \log t$ as $t \to \infty$.*

Remarks. Thus is true for all parameter values. But in the balanced ($\beta > 2\alpha$) case the conjectured exponent $\zeta[I]$ satisfies $\zeta[I] > \alpha$ and so the assertion of the lemma is not sharp.

Proof. Write

$$V_t := \int \log I(y,t)\ dy.$$

Monotonicity of I implies $\log I(y,t+1) - \log I(y,t) \geq 0$. To get a quantitative lower bound on this increment, observe that if the arrival at time $t+1$ joins city i then for $y \in \mathcal{S}(i,t)$ we have $I(y,t+1) = I(y,t)(\frac{N_i(t)+1}{N_i(t)})^\alpha$ and so

$$\log I(y,t+1) - \log I(y,t) = \alpha \log(1 + N_i^{-1}(t)).$$

So

$$E(V_{t+1} - V_t | \mathcal{F}_t) \geq \sum_i p(i,t) a_i(t) \alpha \log(1 + N_i^{-1}(t))$$

where $p(i,t)$ is the conditional probability that the arrival at time $t+1$ joins city i. Now

$$p(i,t) \geq \tfrac{J}{J+1} a_i(t) 1_{(\min_y I(y,t) \geq J)}$$

and so

$$E(V_{t+1} - V_t | \mathcal{F}_t) \geq \frac{\alpha J}{J+1} \sum_i a_i^2(t) \log(1 + N_i^{-1}(t)) \ \times 1_{(\min_y I(y,t) \geq J)}.$$

Choose $n(J)$ such that

$$\log(1 + n^{-1}) \geq \tfrac{J-1}{Jn},\ n \geq n(J)$$

we see

$$E(V_{t+1} - V_t | \mathcal{F}_t) \geq \frac{\alpha(J-1)}{J+1} \sum_{i: N_i(t) \geq n(J)} \tfrac{a_i^2(t)}{N_i(t))} \ \times 1_{(\min_y I(y,t) \geq J)}.$$

Using the inequality (from Cauchy-Schwarz)

$$\sum_i a_i^2/\nu_i \geq (\sum_i a_i)^2/(\sum_i \nu_i)$$

this becomes

$$E(V_{t+1} - V_t|\mathcal{F}_t) \geq \frac{\alpha(J-1)}{t(J+1)} \left(\sum_i a_i(t) 1_{(N_i(t) \geq n(J))} \right)^2 \times 1_{(\min_y I(y,t) \geq J)}.$$

Using Lemmas 5 and 6 we can put this into the format

$$E(V_{t+1} - V_t|\mathcal{F}_t) \geq \tfrac{\alpha(J-1)}{t(J+1)}(1 - \Delta_{J,t})$$

where

$$0 \leq \Delta_{J,t} \leq 1; \quad \lim_t E\Delta_{J,t} = 0.$$

This implies

$$EV_{t+1} - EV_t \geq (\alpha - o(1))/t \text{ as } t \to \infty$$

and the result follows ■

The next lemma shows that, in the unbalanced case, we can relate the presence of large cities to the value of $\int \log I(y,t)\ dy$.

Lemma 8 *Let $0 < \alpha < \infty$ and $\beta < 2\alpha$. Set $\kappa = \frac{\beta}{2}\log(e\pi c_0^{\beta/2})$. Consider an arbitrary configuration at time t.*
(a) $\int \log I(y)\ dy \leq \alpha \log t + \kappa$.
(b) If $\int \log I(y)\ dy \geq \alpha \log t + \kappa - J$ then $N_{(1)} \geq t\exp(-\frac{2}{2\alpha-\beta}J)$.

Proof. By solving for r the equation $c_0 N_i^\alpha r^{-\beta} = x$ we get

$$\text{area } \{y : \text{ influence of city } i \text{ at } y \text{ is } \geq x\ \} \ \leq \pi r^2 = \pi(c_0 x^{-1} N_i^\alpha)^{2/\beta}$$

and so

$$\text{area } \{y : I(y) \geq x\} \leq \pi c_0^{2/\beta} \sum_i N_i^{2\alpha/\beta}\ x^{-2/\beta}.$$

So, setting $A = \pi c_0^{2/\beta}\sum_i N_i^{2\alpha/\beta}$,

$$\begin{aligned} \int \log I(y)\ dy &= \int_0^\infty \text{area } \{y : \ \log I(y) \geq u\}\ du \\ &\leq \int_0^\infty \min(1, A\exp(-2u/\beta)\ du \\ &= \tfrac{\beta}{2}(1 + \log A) \\ &= \kappa + \tfrac{\beta}{2}\log(\sum_i N_i^{2\alpha/\beta}). \end{aligned} \tag{13}$$

Now $2\alpha/\beta > 1$, so the maximum possible value of $\sum_i N_i^{2\alpha/\beta}$ under the constraint $\sum_i N_i = t$ is attained by taking $N_1 = t$. That is, $\sum_i N_i^{2\alpha/\beta} \leq t^{2\alpha/\beta}$ and then (13) gives (a). Alternatively, fix $m < t$ and constrain each N_i to be at most m. Then the maximum possible value of $\sum_i N_i^{2\alpha/\beta}$

is attained by taking $N_1 = N_2 = \ldots = m$ for t/m cities (xxx say more carefully) and so $\sum_i N_i^{2\alpha/\beta} \leq \frac{t}{m}\, m^{2\alpha/\beta}$. In other words,

$$\sum_i N_i^{2\alpha/\beta} \leq t N_{(1)}^{\frac{2\alpha-\beta}{\beta}}.$$

Using (13) and the hypothesis of (b),

$$\alpha \log t + \kappa - J \leq \kappa + \tfrac{\beta}{2} \log t + \tfrac{2\alpha-\beta}{2} \log N_{(1)}$$

which rearranges to the conclusion of (b). ■

Proof of Theorem 2(b) Again write $V_t := \int \log I(y,t)\, dy$. By Lemmas 8(a) and 7

$$V_t \leq \alpha \log t + \kappa; \quad EV_t \geq (\alpha - o(1)) \log t$$

which imply that for fixed $\varepsilon > 0$

$$P(V_t < (\alpha - \varepsilon) \log t) \to 0.$$

Lemma 8(b) now implies that for each $\varepsilon > 0$

$$P(N_{(1)}(t) \geq t^{1-\varepsilon}) \to 0$$

as required. ■

3 The balanced scenario: $\alpha < 1$ and $\beta > 2\alpha$

Recall notation:
$N_i(t) =$ size of city i at time t
$R(y,t) =$ distance from y to nearest city at time t.

We seek to formalize the heuristics in section 1 by studying the following rigorously defined upper and lower exponents for the growth rates of three quantities. In each case we have $\zeta_* \le \zeta^*$.

$\zeta^*[I]$ is the infimum of ϕ such that

$$E\text{area } \{y : I(y,t) \ge t^\phi\} \to 0.$$

$\zeta_*[I]$ is the supremum of ϕ such that

$$E\text{area } \{y : I(y,t) \le t^\phi\} \to 0.$$

$\zeta^*[R]$ is the infimum of ξ such that

$$E\text{area } \{y : R(y,t) \ge t^\xi\} \to 0.$$

$\zeta_*[R]$ is the supremum of ξ such that

$$E\text{area } \{y : R(y,t) \le t^\xi\} \to 0.$$

$\zeta^*[N]$ is the infimum of ψ such that

$$E \sum_i a_i(t) 1_{(N_i(t) \ge t^\psi)} \to 0.$$

$\zeta_*[N]$ is the supremum of ψ such that

$$E \sum_i a_i(t) 1_{(N_i(t) \le t^\psi)} \to 0.$$

In the final two definitions, $a_i(t) = \text{area } \mathcal{S}(i,t)$, and the quantity under consideration is a technically convenient proxy for the more intuitive quantity $t^{-1} \sum_i N_i(t) 1_{(N_i(t) \ge t^\psi)}$ representing proportion of population in large cities (see Lemma 12).

The heuristic arguments suggest the conjectures

$$\zeta_*[I] = \zeta^*[I] = \zeta_*[N] = \zeta^*[N] = \tfrac{\beta}{2-2\alpha+\beta} > 0; \quad \zeta_*[R] = \zeta^*[R] = \tfrac{\alpha-1}{2-2\alpha+\beta} < 0.$$

We summarize our analysis as follows.

*Prop.*9	$\zeta^*[I] \le \alpha\zeta^*[N] - \beta\zeta_*[R]$	definition of $I(y,t)$
*Prop.*11	$\zeta^*[N] \le \frac{\beta - 2\zeta_*[I]}{\beta - 2\alpha}$	I large implies cities can't grow fast
*Prop.*15	$\zeta^*[I] \ge 1 + 2\zeta^*[R]$	I small implies nearby cities founded often
*Prop.*15	$\zeta_*[I] \ge 1 + 2\zeta_*[R]$	I small implies nearby cities founded often
*Prop.*18	$\zeta_*[I] \le 1 + 2\zeta_*[R]$	I large implies nearby cities founded seldom

Note these are all "sharp" in the sense of being equalities for the conjectured values.

xxx obviously this isn't enough to prove the conjectures, but organizes what we know. Maybe helps us to formulate what we need to prove!

xxx the inequalities above imply

$$\zeta_*[I] = 1 + 2\zeta_*[R]$$

and then (using the first two inequalities) we get

$$(\beta^2 - 2\alpha\beta + 4\alpha)\zeta_*[I] + 2(\beta - 2\alpha)\zeta^*[I] \leq \beta^2.$$

Then using $\zeta_*[I] \leq \zeta^*[I]$ we get

$$\zeta_*[I] \leq \tfrac{\beta}{2-2\alpha+\beta}$$

where the right side is the conjectured value.

xxx need 12 inequalities; 3 are trivial; so we have gotten 8 so far!!!!!!

3.1 From the definition of $I(y,t)$

Proposition 9 $\zeta^*[I] \le \alpha\zeta^*[N] - \beta\zeta_*[R]$.

Proof. Write $\sigma(y)$ for the city in whose sphere of influence y resides. Then

$$I(y) = c_0 N^\alpha_{\sigma(y)} |y - x_{\sigma(y)}|^{-\beta} \le c_0 N^\alpha_{\sigma(y)} r^{-\beta}(y).$$

So if ψ and ξ are such that

$$E\text{area}\ \{y : N_{\sigma(y,t)}(t) < t^\psi \text{ and } R(y,t) > t^\xi\} \to 1 \tag{14}$$

then

$$E\text{area}\ \{y : I(y,t) \le c_0 t^{\alpha\psi - \beta\xi}\} \to 1$$

and so $\zeta^*[I] \le \alpha\psi - \beta\xi$. Now because

$$E\text{area}\ \{y : N_{\sigma(y,t)}(t) \ge t^\psi\} = E \sum_i a_i(t) 1_{(N_i(t) \ge t^\psi)},$$

(14) holds when $\xi < \zeta_*[R]$ and $\psi > \zeta^*[N]$, establishing the result. ■

xxx next lemma is similar; not yet used. Its point is that the conjectured exponents satisfy

$$1 = \tfrac{1}{\alpha}\zeta[I] + \tfrac{\beta - 2\alpha}{\alpha}\zeta[R]$$

and so it says that, if I^* is larger than the conjectured value, this cannot be caused by cities *farther away* than the conjectured value $r^*(t)$.

Lemma 10 *Let $\beta > 2\alpha$. For an arbitrary configuration at time t,*

$$\text{area}\ \{y : I(y,t) \ge b \textit{ and } y \in \mathcal{S}(i,t) \textit{ for some } i \textit{ with } |y - x_i| \ge r\} \le \frac{\kappa t}{b^{1/\alpha} r^{\frac{\beta - 2\alpha}{\alpha}}}$$

where $\kappa = \pi c_0^{1/\alpha}$.

Proof. Fix b and r. Let city i have population n_i. Then

$$\text{area}\ \{y :\ |y - x_i| \ge r, I_0(n_i, |y - x_i|) \ge b\} \le \pi(s_i^2 - r^2)^+$$

where s_i is the solution of $c_0 n_i^\alpha s_i^{-\beta} = b$, so $s_i^2 = (c_0 n_i^\alpha b^{-1})^{2/\beta}$. Define n_0 by

$$r^2 = (c_0 n_0^\alpha b^{-1})^{2/\beta}.$$

Then the area under consideration in the lemma (A, say) satisfies

$$\begin{aligned} A &\le \pi \sum_i (s_i^2 - r^2)^+ \\ &\le \pi \sum_{i : n_i \ge n_0} s_i^2 \\ &= \pi c_0^{2/\beta} b^{-2/\beta} \sum_{i : n_i \ge n_0} n_i^{2\alpha/\beta}. \end{aligned}$$

Note the general inequality

$$\sum_i n_i^\gamma \leq \frac{\sum_i n_i}{\min_i n_i^{1-\gamma}},\ 0 < \gamma < 1,\ n_i > 0. \tag{15}$$

Applying this with $\gamma = 2\alpha/\beta$ gives

$$A \leq \pi c_0^{2/\beta} b^{-2/\beta} \times \frac{t}{n_0^{1-2\alpha/\beta}}.$$

Solving the equation for n_0 leads to

$$n_0^{1-2\alpha/\beta} = r^{\frac{\beta-2\alpha}{\alpha}}\ (b/c_0)^{\frac{\beta-2\alpha}{\alpha\beta}}$$

and the result follows. ■

3.2 I large implies cities can't grow fast

xxx idea: if the typical value $I^*(t)$ is large then the sphere of influence of a city is small and so cities can't grow fast. So an lower bound on $I^*(t)$ gives an upper bound on city size.

Proposition 11 *Let $\beta > 2\alpha$. Let $\alpha < \phi < \beta/2$ and suppose*

$$E\text{area } \{y : I(y,t) \leq t^\phi\} \to 0 \text{ as } t \to \infty. \tag{16}$$

Then

$$E\sum_i a_i(t) 1_{(N_i(t)\geq t^\psi)} \to 0. \tag{17}$$

for all ψ satisfying

$$1 > \psi > \frac{\beta - 2\phi}{\beta - 2\alpha}. \tag{18}$$

That is,

$$\zeta^*[N] \leq \frac{\beta - 2\zeta_*[I]}{\beta - 2\alpha}.$$

Proof. The general inequality (15) implies that for each t

$$\sum_i N_i^\gamma(t) 1_{(N_i(t)\geq t^\psi)} \leq \frac{t}{t^{\psi(1-\gamma)}}.$$

Applying this with $\gamma = 2\alpha/\beta$ gives

$$t^{\psi(1-2\alpha/\beta)-1}\ E\sum_i N_i^{2\alpha/\beta}(t) 1_{(N_i(t)\geq t^\psi)} \leq 1.$$

Let $r_i(t)$ be the solution of

$$c_0 N_i^\alpha(t)/r_i^\beta(t) = t^\phi.$$

Then

$$r_i^2(t) = c_0^{1/\beta} N_i^{2\alpha/\beta}(t)\ t^{-2\phi/\beta}.$$

The lower bound on ψ implies $-2\phi/\beta < \psi(1 - 2\alpha/\beta) - 1$ and so

$$E\sum_i r_i^2(t) 1_{(N_i(t)\geq t^\psi)} \to 0. \tag{19}$$

Now let $a_i(t)$ be the area of the sphere of influence of city i and let $a(t) = \text{area } \{y : I(y,t) \leq t^\phi\}$. Then

$$\sum_i a_i(t) 1_{(N_i(t)\geq t^\psi)} \leq a(t) + \sum_i \pi r_i^2(t) 1_{(N_i(t)\geq t^\psi)}$$

the two terms corresponding the the regions y where $I(y,t) < t^\phi$ and where $I(y,t) > t^\phi$. So using (19) and (16)

$$E\sum_i a_i(t) 1_{(N_i(t)\geq t^\psi)} \to 0.$$

■

The next lemma records a relation between $\zeta^*[N]$ and the more intuitive quantity $t^{-1}\sum_i N_i(t) 1_{(N_i(t)\geq t^\psi)}$.

Lemma 12 *For all* $\psi > \zeta^*[N]$,

$$t^{-1} E \sum_i N_i(t) 1_{(N_i(t) \geq t^{\psi})} \to 0.$$

Proof. Take $\psi > \zeta^*[N]$, so by definition

$$E \sum_i a_i(t) 1_{(N_i(t) \geq t^{\psi})} \to 0.$$

Let $q(t+1)$ be the probability that the arrival at time $t+1$ joins a city with population $\geq t^{\psi}$. Then

$$q(t+1) \leq E \sum_i a_i(t) 1_{(N_i(t) \geq t^{\psi})} \to 0$$

the inequality arising from the possibility of founding a new city. Now consider at time t a city with population $N_i(t) \geq 2t^{\psi}$. Then at least half the population arrived at times $s+1 \leq t$ for which $N_i(s+1) \geq t^{\psi} \geq s^{\psi}$, and so

$$E \sum_i \tfrac{1}{2} N_i(t) 1_{(N_i(t) \geq 2t^{\psi})} \leq \sum_{s=1}^{t} q(s) = o(t).$$

This establishes the Proposition, since we may replace ψ by a slightly smaller ψ'. ∎

3.3 I small implies nearby cities founded often

xxx these arguments use $D(y,r)$ = disc of radius r around center y and implicitly assume disc is inside the unit square, so they're not precisely correct as stated in this version, but "essentially correct" because we only care about typical positions y and small r.

xxx study trade-off between influence at a point y and the distance to nearest city. Proposition 15 shows we cannot have both the influence being small and the distance being large.

We start with an easy geometry lemma.

Lemma 13 *Suppose the nearest city to position y is at distance r^*. Then for all $x \in D(y, r^*)$*

$$\left(1 + \tfrac{|x-y|}{r^*}\right)^{-\beta} \leq \frac{I(x)}{I(y)} \leq \left(1 - \tfrac{|x-y|}{r^*}\right)^{-\beta}.$$

Proof. $I(x) = I(x, B)$ for some city B at some position z with $|z-y| \geq r^*$. Since $I(y) \geq I(y, B)$,

$$\frac{I(x)}{I(y)} \leq \frac{I(x,B)}{I(y,B)} \leq \left[\inf_{z:\ |z-y| \geq r^*} \tfrac{|z-x|}{|z-y|}\right]^{-\beta}$$

and the infimum is attained at the point on the boundary closest to x, where the ratio equals $(r^* - |x-y|)/r^*$. For the lower bound, $I(y) = I(y, B)$ for some city B at some position z with $|z - y| \geq r^*$, so

$$\frac{I(x)}{I(y)} \geq \frac{I(x,B)}{I(y,B)} = \left(\tfrac{|z-x|}{|z-y|}\right)^{-\beta} \geq \left(\tfrac{|z-y|+|y-x|}{|z-y|}\right)^{-\beta}$$

and the fraction is maximized when $|z - y| = r^*$. ■

The next result is an immediate consequence, using the rule for founding new cities.

Corollary 14 *Consider an arbitrary configuration at time t and a position y. Let r^* be the distance from y to the nearest city. Define $f(r),\ 0 < r < r^*$ by*

$$f(r)dr = P(\text{arrival at time } t+1 \text{ founds city at distance} \in (r, r+dr) \text{ from } y).$$

Then

$$\tfrac{2\pi r}{1+I(y)}(1 - \tfrac{r}{r^*})^{\beta} \leq f(r) \leq \tfrac{2\pi r}{I(y)}(1 + \tfrac{r}{r^*})^{\beta}.$$

Proposition 15 *Fix a position y. Let $R(y,t)$ be the distance from y to the nearest city. Then*

$$P(I(y,t) \leq b, R(y,t) \geq z) \leq \frac{\kappa(b+1)}{tz^2}, \quad t \geq 1,\ 0 < b, z \tag{20}$$

where $\kappa = 2^{\beta+2}/\pi$. So

$$\zeta^*[I] \geq 1 + 2\zeta^*[R]; \quad \zeta_*[I] \geq 1 + 2\zeta_*[R].$$

Proof. Consider $T := \min\{t : I(y,t) > b \text{ or } R(y,t) < z\}$. On the event $\{T > t\}$ the lower bound in Corollary 14 implies

$$P(r(t+1) \leq \tfrac{z}{2} | \mathcal{F}(t)) \geq \frac{2\pi}{1 + I(y,t)} \int_0^{z/2} r(1 - \tfrac{r}{R(y,t)})^{\beta}\ dr \geq \frac{2\pi}{1+b} \int_0^{z/2} r(\tfrac{1}{2})^{\beta}\ dr = \frac{z^2}{\kappa(b+1)}.$$

But

$$P(T = t+1|\mathcal{F}(t)) \geq P(r(t+1) \leq \tfrac{z}{2}|\mathcal{F}(t)) \text{ on } \{T > t\}$$

and so the distribution of $T - 1$ is stochastically bounded by the Geometric distribution with mean $\kappa(b+1)/z^2$. The event in (20) is $\{T \geq t+1\}$, so inequality (20) follows from Markov's inequality.

Now consider $\xi < \zeta_*[R]$, so

$$E\text{area } \{y : R(y,t) \leq t^\xi\} \to 0.$$

Applying (20) with $z = t^\xi$ shows

$$E\text{area } \{y : I(y,t) \leq b(t)\} \to 0 \text{ whenever } b(t)/t^{1+2\xi} \to 0$$

implying $\zeta_*[I] \geq 1 + 2\xi$ and thus $\zeta_*[I] \geq 1 + 2\zeta_*[R]$. Conversely, consider $\phi > \zeta^*[I]$, so

$$E\text{area } \{y : I(y,t) \geq t^\phi\} \to 0.$$

Applying (20) with $b = t^\phi$ shows

$$E\text{area } \{y : R(y,t) \geq z(t)\} \to 0 \text{ whenever } t^\phi/(tz^2(t)) \to 0$$

implying $\phi \geq 1 + 2\zeta^*[R]$ and thus $\zeta^*[I] \geq 1 + 2\zeta^*[R]$. ■

3.4 I large implies nearby cities founded seldom

xxx continues ideas of previous section.

Lemma 16 *Let $0 < \nu < \mu < 1$. Then there exists a constant $J = J(\nu, \mu)$ such that for each T there exist integers $1 = s_1 \le s_2 \le \ldots \le s_{J+1} = T + 1$ such that*

$$s_j^{-\mu}(s_{j+1} - s_j) \le T^{1-\nu}, \ 1 \le j \le J.$$

Proof. (xxx sketch). Define s_j inductively by $s_j^{-\mu}(s_{j+1} - s_j) = T^{1-\nu}$. Then $s_{j+1} \approx s_j^{\mu} T^{1-\nu}$ and inductively

$$s_j \approx T^{a_j}; \quad a_{j+1} = \mu a_j + (1 - \nu)$$

so that $a_j \to \frac{1-\nu}{1-\mu} > 1$ and we just choose J such that $a_{J+1} > 1$. ■

The next lemma relates exponents for the quantities to exponents for sums of inverses. It uses only the momotonicity property of $I(y, \cdot)$.

Lemma 17 *For all $\phi < \zeta_*[I]$*

$$E\text{area } \{y : \sum_{s=1}^{t} I^{-1}(y, s) \ge t^{1-\phi}\} \to 0.$$

Proof. Take $0 < \phi < \zeta_*[I]$, so by definition

$$E\text{area } \{y : I^{-1}(y, t) \ge t^{-\phi}\} \to 0.$$

Given $\varepsilon > 0$ we can choose $K_\varepsilon < \infty$ such that

$$E\text{area } \{y : I^{-1}(y, t) \ge K_\varepsilon t^{-\phi}\} \le ps, \quad \forall t. \tag{21}$$

Take $0 < \nu < \phi$. Take $J = J(\nu, \phi)$ as in Lemma 16 and for given t take $(1 = s_1, s_2, \ldots, s_{J+1} = t + 1)$ as in that lemma. If

$$I^{-1}(y, s_j) \le K_\varepsilon s_j^{-\nu}, \ 1 \le j \le J$$

then by monotonicity

$$\sum_{s=1}^{t} I^{-1}(y, s) \le \sum_{j=1}^{J} K_\varepsilon s_j^{-\phi}(s_{j+1} - s_j) \le J K_\varepsilon t^{1-\nu}.$$

So using (21)

$$E\text{area } \{y : \sum_{s=1}^{t} I^{-1}(y, s) > J K_\varepsilon t^{1-\nu}) \le J\varepsilon$$

and the result follows. ■

Proposition 18 $\zeta_*[I] \ge 1 + 2\zeta_*[R]$.

Proof. (xxx sketchy). Fix a position y. We want to compare the joint process $(I(y,t), R(y,t))$ with a joint process $(I(y,t), \hat{r}(y,t))$ where $\hat{r}(y,t)$ is defined as follows. Create "virtual points" according to the rule:

given the configuration at time t, place a virtual point at time $t+1$ uniformly in the disc with center y and area $I(y)/2^{\beta}$.

Let $\hat{r}(y,t)$ be the distance from y to the closest virtual point. Using the upper bound in Corollary 14, (xxx details omitted) we can couple $R(y,t)$ and $\hat{r}(y,t)$ such that

$$\hat{r}(y,t) \leq R(y,t).$$

Then

$$\begin{aligned} P(R(y,t) \leq r | I(y,s),\ s \leq t) &\leq P(\hat{r}(y,t) \leq r | I(y,s),\ s \leq t) \\ &\leq 2^{\beta} \pi r^2 \sum_{s=1}^{t} I^{-1}(y,s). \end{aligned}$$

So if $2\phi + q < 0$ then

$$P(R(y,t) \leq t^{\phi},\ \sum_{s=1}^{t} I^{-1}(y,s) \leq t^{q}) \to 0.$$

Now using Lemma 17,

$$E\text{area } \{y : R(y,t) \leq t^{\phi}\} \to 0$$

provided $2\phi + (1 - \zeta_*[I]) < 0$ and the result follows. ■

4 xxx nontrivial geometry

Proposition 19 gives (in the case $\beta > 2\alpha$) a lower bound on the influence function over discs of the size that contains $\Theta(1)$ cities. Here's the essential idea. Suppose there are at least 1 and at most K cities in a disc at time t. If the influence function is small, then at least one city must grow, over the time interval $[t, t+\frac{t}{10}]$, and this growth in population implies a lower bound on the influence function at time $t+\frac{t}{10}$.

Proposition 19 *Let $0 < \alpha < \infty$ and $\beta > 2\alpha$. Let $D(y, 2r)$ be in the unit square. Consider an arbitrary feasible configuration at time t_0. Fix $t_1 > t_0$ and fix b, K. Suppose*

$$\sup\{I(z, t_0) : \ z \in D(y, r/2)\} \geq b. \tag{22}$$

Write $Q(y, t, 2r)$ for the number of cities in $D(y, 2r)$ at time t. Then,

$$P\left(Q(y, t, 2r) \leq q; \quad \inf_{z \in D(y,r)} I(z, t_1) \leq 3^{-\beta} b\right) \leq \frac{\kappa \Delta(q)}{(t_1 - t_0)} \ b^{1/\alpha} r^{\frac{\beta}{\alpha}-2} \tag{23}$$

where κ depends only on the model parameters and $\Delta(q)$ depends only on q.

Note in particular that hypothesis (22) holds for all b if there is some city in $D(y, r/2)$.

xxx in proof below we have written $K(t) = Q(y, t, 2r)$.

We start with a "pure geometry" lemma, not involving our process. Write $D^{(r)}$ for the disc of radius r centered at the origin.

Lemma 20 *Fix $K \geq 1$. Let $D_1, \ldots, D_K$ be discs of arbitrary radii, each centered at some point in $D^{(2)}$. Let $S = \cup_{i=1}^K D_i$ and let $A_i \subseteq D_i$ be such that $(A_i \cap D^{(1)}, 1 \leq i \leq K)$ is a partition of $S \cap D^{(1)}$. Suppose*
(i) S intersects $D^{(1/2)}$;
(ii) S does not cover $D^{(1)}$.
Then

$$\max_{1 \leq i \leq K} \frac{\text{area } (A_i \cap D^{(1)})}{\text{area } (D_i)} \geq \delta(K) \tag{24}$$

where $\delta(K) > 0$ depends only on K.

Note that we may take $\delta(K)$ to be decreasing in K. Our argument gives $\delta(K) \sim K^{-3}$, which presumably could be improved (xxx but improvement would not help of arguments, I guess).

Proof. We may suppose D_1 intersects $D^{(1/2)}$. Consider the connected cluster, say $D_1, \ldots, D_L$ $(1 \leq L \leq K)$ of overlapping discs containing disc D_1. Consider first the case where this cluster lies within $D^{(1)}$. We assert, in this case,

$$\max_{1 \leq i \leq L} \frac{\text{area } (A_i)}{\text{area } (D_i)} \geq L^{-1} \geq K^{-1} \tag{25}$$

establishing (24). If (25) fails then, for each $1 \leq j \leq L$,

$$\text{area } (D_j) = \sum_{i=1}^{L} \text{area } (D_j \cap A_i) \leq \sum_{i=1}^{L} \text{area } (A_i) < L^{-1} \sum_{i=1}^{L} \text{area } (D_i)$$

and summing over j gives a contradiction.

Now consider the alternate case. Here we can choose a sequence $D_1, D_2, \ldots, D_L$ where each successive pair D_i, D_{i+1} overlap and where

$$L = \min\{i : D_i \text{ is not contained in } D^{(1)}\} \text{ satisfies } 1 \leq L \leq K.$$

We assert that

$$\text{area } (S \cap D^{(1)}) \geq \pi(\tfrac{1}{8K})^2. \tag{26}$$

If $L = 1$ then D_1 intersects both $D^{(1/2)}$ and the complement of $D^{(1)}$, so $D_1 \cap D^{(1)}$ contains a disc of diameter $1/2$, implying (26). If $L \geq 2$ and one of $D_1, \ldots, D_{L-1}$ has radius $\geq \frac{1}{8K}$, then (26) is clearly true. If $L \geq 2$ and none of $D_1, \ldots, D_{L-1}$ has radius $\geq \frac{1}{8K}$, then D_{L-1} is contained inside $D^{(3/4)}$ and so D_L intersects both $D^{(3/4)}$ and the complement of $D^{(1)}$, and thus (repeating the argument for D_1 above) area $(D_L \cap D^{(1)}) \geq \pi(1/8)^2$. This establishes (26). Now each D_i has radius at most 3, because it is centered in $D^{(2)}$ and does not cover $D^{(1)}$. So

$$\begin{aligned}
\max_{1 \leq i \leq K} \frac{\text{area } (A_i \cap D^{(1)})}{\text{area } (D_i)} &\geq \frac{\sum_{i=1}^K \text{area } (A_i \cap D^{(1)})}{\sum_{i=1}^K \text{area } (D_i)} \\
&\geq \frac{\text{area } (S \cap D^{(1)})}{K\ \pi 3^2} \\
&\geq \frac{\pi(\frac{1}{8K})^2}{9\pi K}.
\end{aligned}$$

■

The next lemma is a routine martingale-type bound.

Lemma 21 *Let $(N_i(t), 1 \leq i \leq K, t = 1, 2, \ldots)$ be $\{0, 1, 2, \ldots\}$-valued and adapted to a filtration $(\mathcal{F}(t))$. Suppose that, for each t,*
either $N_i(t+1) = N_i(t)\ \forall i$;
or $N_j(t+1) = N_j(t) + 1$ and $N_i(t+1) = N_i(t)\ \forall i \neq j$, for some random j.
Define $q_i(t) = P(N_i(t+1) = N_i(t) + 1 | \mathcal{F}(t))$. Fix a time interval $[t_0, t_1]$ and fix $c > 0$ and $0 < \gamma < 1$. Suppose that, for each $t_0 \leq t < t_1$,

$$q_i(t) \geq c(N_i(t))^\gamma, \text{ for some } i \text{ with } N_i(t) \geq 1. \tag{27}$$

Suppose $\max_i N_i(t_0) \geq 1$. *Then for any integer z,*

$$P(\max_i N_i(t_1) < z) \leq \frac{K^2 z^{1-\gamma}}{(1-\gamma)c(t_1 - t_0)}.$$

Proof. Define $\tau_i(0) = \min\{t \geq t_0 : N_i(t) \geq 1,\ q_i(t) \geq c(N_i(t))^\gamma\}$ and then inductively for $s = 1, 2, 3, \ldots$ define

$$\tau_i(s) = \min\{t > \tau_i(s-1) : q_i(t) \geq c(N_i(t))^\gamma\}.$$

Set $\widetilde{N}_i(s) = N_i(\tau_i(s))$ and $\widetilde{\mathcal{F}}_i(s) = \mathcal{F}(\tau_i(s))$. Then $\widetilde{N}_i(0) \geq 1$ and

$$P(\widetilde{N}_i(s+1) \geq \widetilde{N}_i(s) + 1 | \widetilde{\mathcal{F}}(s)) \geq c(\widetilde{N}_i(s))^\gamma.$$

By comparison with the related pure birth process, $T_i := \min\{t : \widetilde{N}_i(t) \geq z\}$ satisfies

$$ET_i \leq \sum_{j=1}^{z-1} \frac{1}{cj^\gamma} \leq \frac{z^{1-\gamma}}{(1-\gamma)c}.$$

Using Markov's inequality, for any σ

$$P(\widetilde{N}_i(\sigma) < z) = P(T_i > \sigma) \leq \frac{z^{1-\gamma}}{\sigma(1-\gamma)c}$$

and so

$$P(N_i(\tau_i(\sigma)) < z \text{ for some } i) \leq \frac{Kz^{1-\gamma}}{\sigma(1-\gamma)c}.$$

Setting $\sigma = (t_1 - t_0)/K$, the hypothesis (27) implies $\tau_i(\sigma) \leq t_1 - t_0$ for some (random) i, so

$$P(\max_i N_i(t_1) < z) \leq \frac{Kz^{1-\gamma}}{\sigma(1-\gamma)c}.$$

■

xxx some unimportant details slid over in proof above: $\tau_i(s)$ may be ∞; as defined σ not integer.

Proof of Proposition 19 Index the cities in $D(y, 2r)$ at time t by $1 \leq i \leq K(t)$, and write

$$D_i(t) = \{z : I_i(z,t) \geq b\}.$$

Suppose, at some time $t_0 \leq t \leq t_1$, the influence $I(A, z, t)$ at some $z \in D(y, r)$ from some city A outside $D(y, 2r)$ is $\geq b$. Then $\min\{I(A, z, t_1) : y \in D(y, r)\} \geq 3^{-\beta} b$ and so the event in (23) cannot happen. So we may ignore the possibility that the influence within $D(y, r)$ from some city outside $D(y, 2r)$ exceeds b. That is, defining

$$S(t) = \cup_{i=1}^{K(t)} D_i(t),$$

we may assume that $S(t)$ coincides with $\{z : I(z,t) \geq b\}$ on $D(y, r)$. Note that the quantity in (23) can now be bounded as

$$P(K(t_1) \leq K, \ S(t_1) \text{ does not cover } D(y, r)).$$

Write $N_i(t)$ for the population of city i. Let $A_i(t)$ be the "sphere of influence" of city i at time t. We now claim that we can apply Lemma 20 at each time $t_0 \leq t < t_1$. Hypothesis (i) of Lemma 20 follows from hypothesis (22); and if hypothesis (ii) fails we are done. From the conclusion of Lemma 20, at each time $t_0 \leq t < t_1$ there is a random i such that

$$\text{area } (A_i(t)) \geq \delta(K) \text{area } (D_i(t)).$$

(xxx notes: we have scaled Lemma 20 by r; and we may assume $K_i(t) \leq K$ or we are done). From the explicit form of the influence function

$$I_0(m, r) = c_0 m^\alpha r^{-\beta}$$

we calculate

$$\text{area } (D_i(t)) = \pi (c_0 b^{-1} N_i^\alpha(t))^{2/\beta}.$$

So (xxx ignoring possibility that a new arrival founds a new city) condition (27) holds with

$$c = \delta(K) \pi (c_0 b^{-1})^{2/\beta}, \quad \gamma = 2\alpha/\beta$$

and by assumption we are in the case $0 < \gamma < 1$. Now define w by

$$c_0 w^\alpha (3r)^{-\beta} = b.$$

At time t_1, if some city in $D(y, 2r)$ has size at least w, then its influence is at least b over all of $D(y, r)$. So by the conclusion of Lemma 21

$$\begin{aligned} P(K(t_1) \le K,\ S(t_1) \text{ does not cover } D(y,r)) &\le P(K(t_1) \le K,\ \max_i N_i(t_1) < w) \\ &\le \frac{K^2 w^{1-\gamma}}{(1-\gamma)c(t_1 - t_0)} \end{aligned}$$

and this bound reduces to the form stated in (23). ■

5 The critical case

xxx haven't tried to study the unbalanced case yet ......

Here we take $\alpha = 1$ and $\beta > 2$.

Heuristics Looking at the heuristic argument in section 1, equation (4) becomes

$$\tfrac{dM}{dt} \approx t^{-1} M^{-\beta/2+1}$$

whose solution is

$$M(t) \approx \log^{2/\beta} t.$$

Continuing the heuristics as before gives

$$I^*(t) \approx N^*(t) \approx t \log^{1-\frac{2}{\beta}} t; \quad R^*(t) \approx \log^{-1/\beta} t.$$

So in particular we may conjecture the following "just sublinear" growth rate for the largest city.

Conjecture 22 *If $\alpha = 1$ and $\beta > 2$ then $N_{(1)}(t) = t \log^{1-\frac{2}{\beta} \pm o(1)} t$ a.s.*

A natural first step would be to prove $t^{-1} N_{(1)}(t) \to 0$ a.s.. In this draft we have managed only to prove the "lim inf" version.

Theorem 23 *If $\alpha = 1$ and $\beta > 2$ then $\liminf_t t^{-1} N_{(1)}(t) = 0$ a.s.*

Remarks. Not only is the result unsatisfactory, but the proof "by contradiction" is complicated. Hopefully one can use these ingredients to find a better proof of a better result! The proof uses the "geometry" result, Proposition 19.

Proof of Theorem 23 To start the proof by contradiction, we take the negation of "$\liminf_t t^{-1} N_{(1)}(t) = 0$ a.s." to be (xxx not precise, but fix-able)

$$\exists a' \text{ such that } N_{(1)}(s) \geq a's \ \forall s.$$

This implies there exists c_1 such that

$$\min_y I(y, s) \geq c_1 s \ \forall s. \tag{28}$$

We now embark upon a series of lemmas.

The first lemma uses a more delicate analysis of the interplay between $R(y, t)$ and $I(y, t)$ given in Corollary 14. The main idea in its proof is to compare $R(y, t)$ to the process on states $\{2^{-j}\}$ which transitions from 2^{-j} to 2^{-j-1} at rate 2^{-2j}. We will apply with $r = O(\log^{-1/2} t)$ and $q = O(1)$, making the bound small.

Lemma 24 *Let $\alpha = 1$ and $\beta > 2$. Suppose (28) holds, where we may take $c_1 < 1$. Take constants $0 < c_2, r < 1 < K < \infty$, a position y and a time t_0, and an integer $q \geq 1$. Let $Q(y, t_1, 2r)$ be the number of cities at time $t_1 > t_0$ within distance $2r$ from y. Then*

$$P\left(\frac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \leq Ks)} > c_2 \text{ and } \{R(y, t_0) > r/2 \text{ or } Q(y, t_1, 2r) \geq q\} \right)$$

$$\leq \frac{4\pi r^2 (1 + \log t_1)}{q c_1} + C \exp\left(-\frac{2^{-3} r^2 (c_2 \log t_0 - \frac{7}{2} + \log_2 r)}{\kappa K} \right) \tag{29}$$

where κ depends only on β and C is the numerical constant defined at (36).

Proof. The expected density of cities at position x at time t equals

$$E(1 + \sum_{s=1}^{t-1} \frac{1}{1 + I(x,s)}) \leq 1 + \sum_{s=1}^{t-1} \frac{1}{1 + c_1 s} \leq 1 + \int_0^t \frac{1}{1 + c_1 s}\ ds = 1 + c_1^{-1} \log(c_1 t) \leq c_1^{-1}(1 + \log t)$$

the final inequality because $c_1 < 1$. So

$$EQ(y, t, 2r) \leq c_1^{-1}(1 + \log t) \times \pi(2r)^2$$

and Markov's inequality gives

$$P(Q(y, t, 2r) \geq q) \leq c_1^{-1}(1 + \log t)\pi(2r)^2/q$$

which (for $t = t_1$) is the first term in (29). Thus it remains to show

$$P\left(\frac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \leq Ks)} > c_2 \text{ and } R(y, t_0) > r/2 \right) \leq C \exp\left(-\frac{2^{-3} r^2 (c_2 \log t_0 - \frac{5}{2} + \log_2 r)}{\kappa K} \right). \tag{30}$$

Write $R(s) = R(y, s)$. Using the lower bound in Corollary 14, (xxx provided disc in square)

$$P(R(s+1) \leq \tfrac{1}{2} R(s) | \mathcal{F}(s)) \geq \kappa^{-1} \frac{R^2(s)}{1 + I(y,s)} \tag{31}$$

for $\kappa^{-1} = \int_0^{1/2} 2\pi r(1-r)^\beta\ dr$. Define stopping times

$$H_j = \min\{u : R(u) \leq 2^{\frac{3}{2} - j}\}, \quad j = 1, 2, 3, \ldots$$

where $H_1 = 1$ because $R(1) \leq \sqrt{2}$. Define stopping times $\tau_1 = 0$ and

$$\tau_j = \min\{u > \tau_{j-1} : \sum_{s=\tau_{j-1}+1}^{u} s^{-1} 1_{(1+I(y,s) \leq Ks)} \geq w_j - w_{j-1}\}$$

where $0 = w_1 < w_2 < w_3 < \ldots$ are constants to be defined later. Because $\frac{1}{1+I(y,s)} \geq \frac{1}{Ks}$ on $\{1 + I(y,s) \leq Ks\}$, this definition implies

$$\sum_{s=\tau_{j-1}+1}^{\tau_j} \frac{1}{1 + I(y,s)} \geq \frac{w_j - w_{j-1}}{K}. \tag{32}$$

We seek to upper bound $P(H_j > \tau_j + 1)$, inductively on j. Fix j and u and consider the event $\{\tau_j = u-1, 2^{\frac{1}{2}-j} < R(u) \le 2^{\frac{3}{2}-j}\}$. On this event

$$\begin{aligned}
P(H_{j+1} > \tau_{j+1}+1|\mathcal{F}(u)) &= P(R(\tau_{j+1}+1) > 2^{\frac{1}{2}-j}|\mathcal{F}(u)) \\
&\le P(R(s+1) \ge \tfrac{1}{2}R(s),\ R(s) > 2^{\frac{1}{2}-j},\ \forall u \le s \le \tau_{j+1}|\mathcal{F}(u)) \\
&= E\left(\prod_{s=u}^{\tau_{j+1}} P(R(s+1) \ge \tfrac{1}{2}R(s)|\mathcal{F}(s))\ 1_{(R(s)>2^{\frac{1}{2}-j})}\middle|\mathcal{F}(u)\right) \\
&\le E\left(\prod_{s=u}^{\tau_{j+1}} \left(1-\kappa^{-1}\frac{R^2(s)}{1+I(y,s)}\right) 1_{(R(s)>2^{\frac{1}{2}-j})}\middle|\mathcal{F}(u)\right) \text{ by (31)} \\
&\le E\left(\prod_{s=u}^{\tau_{j+1}} \left(1-\kappa^{-1}\frac{2^{1-2j}}{1+I(y,s)}\right) 1_{(R(s)>2^{\frac{1}{2}-j})}\middle|\mathcal{F}(u)\right) \\
&\le E\left(\exp\left(-\kappa^{-1}2^{1-2j}\sum_{s=u}^{\tau_{j+1}}\frac{1}{1+I(y,s)}\right)\middle|\mathcal{F}(u)\right) \\
&\le \exp\left(-\frac{2^{1-2j}(w_{j+1}-w_j)}{\kappa K}\right) \text{ by (32) .}
\end{aligned}$$

Now consider the event $\{\tau_j = u-1, H_j \le u\}$. On this event, $R(u) \le 2^{\frac{3}{2}-j}$. If $R(u) \le 2^{\frac{1}{2}-j}$ then $H_{j+1} \le u \le \tau_{j+1}$, and so the previous bound implies

$$P(H_{j+1} > \tau_{j+1}+1|\mathcal{F}(u)) \le \exp\left(-\frac{2^{1-2j}(w_{j+1}-w_j)}{\kappa K}\right) \text{ on } \{\tau_j = u-1, H_j \le u\}.$$

This can be rewritten as

$$P(H_{j+1} > \tau_{j+1}+1|\mathcal{F}(\tau_j+1)) \le \exp\left(-\frac{2^{1-2j}(w_{j+1}-w_j)}{\kappa K}\right) \text{ on } \{H_j \le \tau_j+1\}.$$

By induction (xxx details)

$$P(H_J > \tau_J + 1) \le \sum_{j=1}^{J-1} \exp\left(-\frac{2^{1-2j}(w_{j+1}-w_j)}{\kappa K}\right). \tag{33}$$

To relate this to (30), note first that the definition of (τ_j) implies

$$\sum_{s=1}^{\tau_J} s^{-1} 1_{(1+I(y,s)\le Ks)} \le w_J + J$$

the final "+1" term from overshoot at each stage. Suppose we choose (w_j) and J such that

$$w_J + J \le c_2 \log t_0. \tag{34}$$

Then

$$\{\tfrac{1}{\log t_0}\sum_{s=1}^{t_0} s^{-1}1_{(1+I(y,s)\le Ks)} > c_2\} \text{ implies } \{\tau_J < t_0\}$$

and $\{R(t_0) > 2^{\frac{3}{2}-J}\}$ implies $\{H_J \ge t_0+1\}$. So

$$\begin{aligned}
P(\tfrac{1}{\log t_0}\sum_{s=1}^{t_0} s^{-1}1_{(1+I(y,s)\le Ks)} > c_2 \text{ and } R(y,t_0) > 2^{\frac{3}{2}-J}) &\le P(\tau_J < t \text{ and } H_J \ge t+1) \\
&\le P(H_J > \tau_J + 1). \qquad (35)
\end{aligned}$$

Given J, set $w_1 = 0$ and

$$w_{j+1} - w_j = (c_2 \log t - J)2^{j-J}, \ 1 \le j \le J-1$$

so that $w_j = \sum_{j=1}^{J-1}(w_{j+1} - w_j) < (c_2 \log t - J)$ and hence (34) is satisfied. Note for future reference that there exists a numerical constant C such that

$$\min\left(1, \sum_{i=0}^{\infty} \exp(-a2^i)\right) \le C \exp(-a), \ 0 < a < \infty. \tag{36}$$

Combining (35) and (33),

$$\begin{aligned} P(\tfrac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \le Ks)} > c_2 \text{ and } R(y,t_0) > 2^{\frac{3}{2}-J}) \ &\le \ \sum_{j=1}^{J-1} \exp\left(-\frac{2^{1-j-J}(c_2 \log t_0 - J)}{\kappa K}\right) \\ &= \ \sum_{i=0}^{J-2} \exp(-a2^i) \end{aligned}$$

where we have substituted $i = J - 1 - j$ and set

$$a = \frac{4(c_2 \log t_0 - J)2^{-2J}}{\kappa K};$$

so now

$$P(\tfrac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \le Ks)} > c_2 \text{ and } R(y,t_0) > 2^{\frac{3}{2}-J}) \le C \exp(-a).$$

Given r, let J be the smallest integer such that $2^{\frac{3}{2}-J} \le r/2$; so $2^{\frac{3}{2}-J} > 2^{-2}r$ and so

$$2^{-2J} > 2^{-7}r^2; \quad -J > \log_2 r - \tfrac{7}{2}$$

giving finally

$$P(\tfrac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \le Ks)} > c_2 \text{ and } R(y,t_0) > r/2) \le C \exp\left(-\frac{2^{-3}r^2(c_2 \log t_0 - \frac{7}{2} + \log_2 r)}{\kappa K}\right)$$

which is (30). ∎

Lemma 25 *Under the hypotheses of Lemma 24, for $K \le B < \infty$,*

$$P\left(\tfrac{1}{\log t_0} \sum_{s=1}^{t_0} s^{-1} 1_{(1+I(y,s) \le Ks)} > c_2 \text{ and } I(y,t_1) \le Bt_1)\right) \to 0$$

uniformly over the range $t_0 \to \infty, t_1 \to \infty, 1 + D^{-1} \le \frac{t_1}{t_0} \le D$ for fixed $1 < D < \infty$.

Proof. This is basically just a matter of identifying the conclusions of Lemma 24 with the hypotheses of Proposition 19. In fact Proposition 19 implies that for any b

$$P\left(Q(y,t_1,2r) \le q; \quad I(y,t_1) \le 3^{-\beta} b | \mathcal{F}(t_0)\right) \le \frac{\kappa \Delta(q)}{(t_1 - t_0)} \ b \ r^{\beta - 2} \tag{37}$$

on the event $\{R(y,t_0) \le r/2\}$. We apply this inequality with b defined by $3^{-\beta}b = Bt_1$. Combining with Lemma 24 (the logic here is abstracted in Lemma 26 below) we conclude that

$$P\left(\tfrac{1}{\log t_0}\sum_{s=1}^{t_0} s^{-1}1_{(1+I(y,s)\le Ks)} > c_2 \text{ and } I(y,t_1) \le Bt_1)\right)$$

$$\le \frac{4\pi r^2(1+\log t_1)}{qc_1} + C\exp\left(-\frac{2^{-3}r^2(c_2\log t_0 - \frac{7}{2} + \log_2 r)}{\kappa K}\right) + \frac{\kappa\Delta(q)}{(t_1-t_0)}\ (3^\beta Bt_1)r^{\beta-2}.$$

Here we may choose $0 < r < 1$ and integer $q \ge 1$. To obtain asymptotics (xxx very messy to say this precisely, so I haven't tried!) first set $r = w\log^{-1/2} t_0$ for large w, and the terms are essentially

$$O(w^2/q) + O(\exp(-\kappa w^2)) + O\left(\Delta(q)\tfrac{t_1}{t_1-t_0}\log^{-(\beta-2)/2} t_0\right).$$

By taking $q = q(t_0) \uparrow \infty$ very slowly we make the first and third terms tend to 0; finally we can take w arbitrarily large. ■

The logic used to combine (37) with Lemma 24 can be abstracted as follows; the proof is straightforward.

Lemma 26 *Let $A_1(t_0), A_2(t_0)$ be arbitrary events in $\mathcal{F}(t_0)$ and let $A_3(t_1), A_4(t_1)$ be arbitrary events in $\mathcal{F}(t_1)$, where $t_1 > t_0$. If*

$$P(A_1(t_0) \text{ and } \{A_2(t_0) \text{ or } A_3(t_1)\}) \le a_1$$

$$P(A_3^c(t_1) \text{ and } A_4(t_1)|\mathcal{F}(t_0)) \le a_2 \text{ on } A_2^c(t_0)$$

then $P(A_1(t_0)$ and $A_4(t_1)) \le a_1 + a_2$.

To continue, we isolate the following lemma, perhaps suitable for a homework problem in a graduate probability course.

Lemma 27 *Let $S_n = \sum_{i=1}^n X_i$ where $0 \le X_i \le 1$. Suppose that for each $\varepsilon > 0$*

$$E(X_{n+1}-\varepsilon)^+1_{(S_n > \varepsilon n)} \to 0.$$

Then $n^{-1}S_n \to 0$ in probability.

Proof. Fix ε and consider $T := \max\{m \le n : S_m \le \varepsilon m\} \ge 0$. On the event $\{S_n \ge 2\varepsilon n\}$ we have

$$2\varepsilon n - \varepsilon T \le S_n - S_T = \sum_{i=T=1}^{n} X_i \le (n-T)\varepsilon + \sum_{i=T+1}^{n}(X_i-\varepsilon)^+.$$

This implies

$$\varepsilon n \le 1 + \sum_{i=1}^{n-1}(X_{i+1}-\varepsilon)^+1_{(S_i>\varepsilon i)} \text{ on } \{S_n \ge 2\varepsilon n\}.$$

Taking expectations,

$$\varepsilon nP(S_n \ge 2\varepsilon n) \le 1 + \sum_{i=1}^{n-1}E(X_{i+1}-\varepsilon)^+1_{(S_i>\varepsilon i)} = o(n)$$

by hypothesis, so $P(S_n \ge 2\varepsilon n) \to 0$ as required. ■

Lemma 28 *Let $\alpha = 1$ and $\beta > 2$, and suppose (28) holds. Then*

$$\lim_t \tfrac{1}{\log t} \sum_{s=1}^{t} s^{-1} E \text{area } \{y : I(y,s) \leq Ks\} = 0 \textit{ for each } K < \infty. \tag{38}$$

Proof. Fix $B = K$ and c_2 and a position y, and set

$$X_i = \sum_{s=2^{i-1}+1}^{2^i} s^{-1} 1_{(1+I(y,s) \leq Ks)}.$$

From the conclusion of Lemma 25 we may apply Lemma 27 (xxx details omitted; really need to put in small gap and sum over $[(1+\eta)2^{i-1}, 2^i]$) to show

$$\tfrac{1}{\log t} \sum_{s=1}^{t} s^{-1} 1_{(1+I(y,s) \leq Ks)} \to 0 \text{ in probability.}$$

Taking expectations and integrating over y gives (38) . ■

Lemma 28 and the next lemma now complete the "proof by contradiction" of Theorem 23. Recall we are seeking to prove (40) by assuming it false, which implies (28), so then Lemmas 28 and 29 imply (40) true. Note however that, because of this peculiar proof structure, we have not actually proved (39) to be true!

Lemma 29 *Let $\alpha = 1$ and $\beta > 2$. Suppose (28) and (38) hold. Then*

$$\lim_t \tfrac{1}{\log t} \sum_{s=1}^{t} s^{-1} E(s^{-1} N_{(1)}(s)) = 0 \tag{39}$$

and in particular

$$\liminf_t t^{-1} N_{(1)}(t) = 0 \textit{ a.s.} \tag{40}$$

Remark The lemma remains true if we omit the logarithmic averaging in both hypothesis (38) and conclusion (39).

Proof. Set $\gamma = 2 - \frac{2}{\beta}$, so $1 < \gamma < 2$, and consider

$$S(t) := \sum_i N_i^\gamma(t).$$

From the process dynamics,

$$E(S(t+1) - S(t)|\mathcal{F}(t)) \leq 1 + \sum_i a_i(t) \left((N_i(t)+1)^\gamma - N_i^\gamma(t)\right)$$

where the 1 bounds the contribution from possible founding of a new city. Taking κ such that $(n+1)^\gamma - n^\gamma \leq \kappa n^{\gamma-1}$ for all $n \geq 1$,

$$E(S(t+1) - S(t)|\mathcal{F}(t)) \leq 1 + \kappa \sum_i a_i(t) N_i^{\gamma-1}(t). \tag{41}$$

Fix K and write

$$\begin{aligned} A_K(t) &= \text{area } \{y : I(y,t) \le Kt\} \\ A_{K,i}(t) &= \text{area } \{y \in \mathcal{S}(i,t) : I(y,t) \le Kt\}. \end{aligned}$$

Now

$$\text{area } \mathcal{S}(i,t) \le \pi r^2 + A_{K,i}(t) \text{ where } r \text{ solves } c_0 N_i(t) r^{-\beta} = Kt$$

and so

$$a_i(t) \le \pi \left(\tfrac{c_0 N_i(t)}{Kt}\right)^{2/\beta} + A_{K,i}(t).$$

Inserting into (41), we can bound by the case where the largest city i has $A_{K,i}(t) = A_K(t)$, and so

$$E(S(t+1) - S(t)|\mathcal{F}(t)) \le 1 + \kappa A_K(t) N_{(1)}^{\gamma-1}(t) + \kappa\pi \left(\tfrac{c_0}{Kt}\right)^{2/\beta} \sum_i N_i^{2/\beta}(t) N_i^{\gamma-1}(t).$$

We chose γ to make $\frac{2}{\beta} + \gamma - 1 = 1$, so the final sum equals t; crudely bounding $N_{(1)}(t)$ by t in the other term leads to

$$E(S(t+1) - S(t)|\mathcal{F}(t)) \le 1 + \kappa t^{\gamma-1} A_K(t) + \kappa\pi (c_0/K)^{2/\beta}\ t^{\gamma-1}.$$

Now take expectation, multiply by $t^{-\gamma}$ and sum:

$$\tfrac{1}{\log t} \sum_{s=1}^t s^{-\gamma}(ES(s+1) - ES(s)) \le O(\tfrac{1}{\log t}) + \tfrac{\kappa}{\log t} \sum_{s=1}^t s^{-1} EA_K(s) + O(K^{-2/\beta}).$$

Taking K arbitrarily large and using hypothesis (**??**), the right side is $o(1)$, so summing by parts gives

$$\tfrac{1}{\log t} \sum_{s=1}^t s^{-\gamma-1} ES(s) = o(1).$$

For fixed $\varepsilon > 0$,

$$E(s^{-1} N_{(1)}(s)) \le \varepsilon + P(N_{(1)}(s) \ge \varepsilon s) \le \varepsilon + (\varepsilon s)^{-\gamma} ES(s)$$

and so

$$\tfrac{1}{\log t} \sum_{s=1}^t s^{-1} E(s^{-1} N_{(1)}(s)) \le \varepsilon + o(1)$$

giving (39). Now note that for any real sequence (a_s) we have

$$\liminf_t a_t \le \liminf_t \tfrac{1}{\log t} \sum_{s=1}^t s^{-1} a_s.$$

So by Fatou's lemma

$$\begin{aligned} E \liminf_t t^{-1} N_{(1)}(t) &\le \liminf_t E(t^{-1} N_{(1)}(t)) \\ &\le \liminf_t \tfrac{1}{\log t} \sum_{s=1}^t s^{-1} E(s^{-1} N_{(1)}(s)) \\ &= 0. \end{aligned}$$

■

6 Ideas not used

6.1 xxx

xxx idea: if $I^*(t)$ too small then new cities appear locally and increase it

xxx seek to argue $\zeta[I] \geq \alpha - (\beta - 2\alpha)\zeta[R]$

Proposition 30 *Fix a position y and time t_0, and consider an arbitrary configuration at time t_0. Suppose at time t_0 the closest point to y is at distance r^*. Suppose at time $t_0 + 1$ a new city A is founded at distance $r < r^*$ from y. Let $t_1 > 0$. Then*

$$P(I(y, t_0 + t_1) < b, \text{ no other city in } D(y, r^*) \text{ at time } t_0 + t_1) = o(1)$$

for

$$b = \frac{c\,[(1 - 2\alpha/\beta)\pi t_1 (r^* - r)/(4r^*)]^\alpha}{(r^*)^{\beta - 2\alpha}}. \tag{42}$$

xxx vague about meaning of $o(1)$.

Proof. Set

$$\varepsilon = \tfrac{r^* - r}{2r^*}.$$

Write $J(\cdot)$ for influence function when we exclude the influence of city A. Note that for $x \in D(y, (1-\varepsilon)r^*)$ and $z \notin D(y, r^*)$ we have $|z - x|/|z - y| \geq \varepsilon$ and hence, if there are no other cities inside $D(y, r^*)$.

$$\max\{J(x)/J(y) : x \in D(y, (1-\varepsilon)r^*)\} \leq \varepsilon^{-\beta}$$

Thus we may assume (xxx explain)

$$J(x) \leq b\varepsilon^{-\beta}, \quad x \in D(y, (1-\varepsilon)r^*).$$

Let $N(t)$ be the population of city A at time $t_0 + t$. Define n_0 by

$$c_0 n_0^\alpha / (\varepsilon r^*)^\beta = b\varepsilon^{-\beta}.$$

As long as $N(t) \leq n_0$, the sphere of influence of city A contains the disc, centered at city A, of radius $r(t) \leq \varepsilon r^*$ given by

$$c_0 N^\alpha(t)/r^\beta(t) = b\varepsilon^{-\beta}.$$

So, as long as $N(t) \leq n_0$,

$$P(N(t+1) = N(t) + 1 | N(t)) \geq \pi r^2(t) = \pi\varepsilon (c_0/b)^{2/\beta} N^{2\alpha/\beta}(t).$$

Temporarily write the right side as $cN^\gamma(t)$. Consider $T := \min\{t : N(t) = n_0\}$. By comparison with the related pure birth process,

$$ET \leq \sum_{j=1}^{n_0 - 1} \frac{1}{cj^\gamma} \leq \frac{n_0^{1-\gamma}}{(1-\gamma)c}.$$

By calculating the variance of the related pure birth process and using Chebyshev's inequality (xxx details omitted)

$$P\left(T > \frac{2n_0^{1-\gamma}}{(1-\gamma)c}\right) = o(1)$$

xxx vague about sense of limit.

From the definitions of n_0 and c

$$\frac{2n_0^{1-\gamma}}{(1-\gamma)c} = \frac{2}{(1-2\alpha/\beta)\pi\varepsilon}(b/c_0)^{1/\alpha}(r^*)^{\frac{\beta-2\alpha}{\alpha}}.$$

The choice (42) for b makes this equals t_1, so

$$P(N(t_1) < n_0) = P(T > t_1) = o(1).$$

But when $N(t_1) \geq n_0$ the influence of city A at t is at least

$$c_0 n_0^\alpha/(r^*)^\beta = b$$

and so $I(y, t_0 + t_1) \geq b$. ■

6.2 xxx

Heuristics. Let's indicate one way to try to use Proposition 19. Choose $r(t)$ such that at time $\approx t$ a typical disc of radius $\approx r(t)$ is likely to have at least 1 but at most $O(1)$ vertices. Apply Proposition 19: we get a lower bound

$$P(\min\{I(y,t) : y \text{ in typical radius-}r \text{ disc}\} \leq b(t)) \text{ is small}$$

where $b(t)$ solves

$$\frac{b^{1/\alpha} r^{\frac{\beta}{\alpha}-2}}{t} \approx 1; \quad \text{so } b(t) \approx t^{\alpha}(r(t))^{-(\beta-2\alpha)}.$$

If this were a lower bound over the entire square then we would have an upper bound on the rate of growth of $M(t)$ = number of cities: $dM(t)/dt \leq 1/b(t)$. Suppose we can show this. Then suppose $M(t) \approx t^{\theta}$ for some θ. A disc of radius $r(t) = t^{(1-\theta)/2}$ has mean $O(1)$ cities; suppose we can show enough non-clustering that such discs are likely to have at least one city. Then, since $dM(t)/dt \approx t^{\theta-1}$, we get the inequality

$$\theta - 1 \leq -\alpha + (\beta - 2\alpha)(1-\theta)/2$$

which becomes

$$\theta \leq \frac{1 - 2\alpha + \beta/2}{1 - \alpha + \beta/2}.$$

This is consistent with the previous heuristic which gave

$$\theta = \frac{1-\alpha}{1-\alpha+\beta/2}.$$

6.3 xxx

Lemma 31 *Let* $1 = W_1 \geq W_2 \geq \ldots > 0$ *be adapted to* $(\mathcal{F}_t)$*, and suppose*

$$E(W_{t+1} - W_t | \mathcal{F}_t) \leq -\tfrac{aW_t}{t}$$

for fixed $a > 0$*. Then*

$$P(W_t \geq t^{\varepsilon - a}) \to 0; \qquad \text{all } \varepsilon > 0.$$

Proof. Set $V_{t+1} = \frac{W_t - W_{t+1}}{W_t}$, so that $0 \leq V_{t+1} < 1$ and $E(V_{t+1}|\mathcal{F}_t) \geq a/t$. Then

$$W_t = \prod_{s=2}^{t} \frac{W_s}{W_{s-1}} = \prod_{s=2}^{t} (1 - V_s) \leq \exp(-\sum_2^t V_s) \leq \exp(-\sum_2^t \tilde{V}_s) \tag{43}$$

where $0 \leq \tilde{V}_s \leq V_s$ is constructed such that $E(\tilde{V}_{s+1}|\mathcal{F}_s) = a/s$. Now

$$\text{var}\ (\sum_1^{t-1} \tilde{V}_{s+1}) \leq \sum_1^{t-1} \tfrac{a}{s}(1 - \tfrac{a}{s}) \leq \sum_1^{t-1} \tfrac{a}{s} = E(\sum_1^{t-1} \tilde{V}_{s+1}) \sim a \log t$$

and Chebyshev's inequality implies that for $\varepsilon > 0$

$$P(\sum_1^{t-1} \tilde{V}_{s+1} \leq (a - \varepsilon) \log t) \to 0 \text{ as } t \to \infty.$$

Applying (43) gives the result. ■

6.4 The associated dynamic system

Suppose we modify the model by starting with n cities in given positions $x_1, \ldots, x_n \in [0,1]^2$, and then not allowing any new cities to be founded. Writing $N_i(t)$ for city i population at time t in this setting, it is intuitively clear that normalized and time-changed populations $Z_i(t) = e^{-t} N_i(e^t)$ should approximate the following deterministic process which we call the *associated dynamic system* $(z_i(t), 1 \le i \le n)$. Its state space is

$$\{(z_1, \ldots, z_n) : \ z_i \ge 0, \sum_i z_i = 1\}$$

and its dynamics are

$$\frac{dz_i}{dt} = -z_i + \text{area } \mathcal{S}_i(z_1, \ldots, z_n)$$

where $\mathcal{S}_i$ is the "sphere of influence of city i" defined in essentially the same way as before:

$$\mathcal{S}_i(z_1, \ldots, z_n) = \{y \in [0,1]^2 : \ z_i^\alpha |y - x_i|^{-\beta} = \max_j z_j^\alpha |y - x_j|^{-\beta}\}.$$

In the following conjecture, we suppose the city positions (x_i) and the initial values $z_i(0)$ are "in general position" and that each $z_i(0) > 0$.

Conjecture 32 *(a) If $\alpha > 1$ or $0 < \beta < 2\alpha$ then there exists an i such that $z_i(t) \to 1$.*
(b) If $0 < \alpha < 1$ and $\beta > 2\alpha$ then $z_i(t) \to z_i(\infty)$, the limit not depending on initial values and satisfying $z_i(\infty) > 0 \ \forall i$.

xxx relate to our stochastic model.

In this connection we mention the notion of *centroidal Voronoi tessellation* [3], an unweighted tessellation in which each point is the centroid (center of mass) of its cell. Such tessellations are obviously the fixed points of a dynamical system in which points (x_i) are moved toward the centroids of their cells.

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