Hammersley's Interacting Particle Process and Longest Increasing Subsequences

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Abstract

In a famous paper [8] Hammersley investigated the length L_n of the longest increasing subsequence of a random n-permutation. Implicit in that paper is a certain one-dimensional continuous-space interacting particle process. By studying a hydrodynamical limit for Hammersley's process we show by fairly "soft" arguments that $\lim n^{-1/2} EL_n = 2$. This is a known result, but previous proofs (Vershik - Kerov [14]; Logan - Shepp [11]) relied on hard analysis of combinatorial asymptotics.

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1 Introduction

An *increasing subsequence* $i_1, i_2, \ldots, i_k$ of a permutation $i \to \pi(i)$ is a subsequence such that

$$i_1 < i_2 < \ldots < i_k; \quad \pi(i_1) < \pi(i_2) < \ldots \pi(i_k).$$

For instance, the permutation

$$7 \ 2 \ 8 \ 1 \ 3 \ 4 \ 10 \ 6 \ 9 \ 5 \qquad (1)$$

(for which $\pi(1) = 4, \pi(2) = 2, \pi(3) = 5, \ldots$) has an increasing subsequence

$$1 \ 3 \ 4 \ 6 \ 9 \qquad (2)$$

of length 5, which is the longest possible for that permutation. Write L_n for the length of the longest increasing subsequence of a uniform random permutation of $\{1, 2, \ldots, n\}$. Traditional motivation for studying L_n is given in [4, 5, 8], but the modern bottom line is that L_n provides an entry to a rich and diverse circle of mathematical ideas (see [1]). Vershik - Kerov [14] and Logan - Shepp [11] relied on hard analysis of asymptotics of random Young tableaux to prove $EL_n \sim 2n^{1/2}$. The purpose of this paper is to point out that this result follows from a hydrodynamical limit theorem for a certain interacting particle process which we name *Hammersley's process*. This is a *continuous*-space process, informally specified as follows. At each time there is a locally finite configuration of particles on R^+. There is a space-time Poisson process of "events"; when an event occurs at position x, the nearest particle to the right of x is moved to position x.

Sections 2.1 and 2.2 relate the limit constant $c \equiv \lim_n n^{-1/2} EL_n$ to a limit constant involving Hammersley's process (the ideas in these sections are implicit in Hammersley [8]). The main result, Theorem 5, is stated in section 2.3, where we also give the heuristic hydrodynamical argument that $c = 2$, and outline the structure of the proof, which occupies the remainder of section 2. Our results for Hammersley's process are in many ways analogous to standard results for the (simple, completely asymmetric) *exclusion process*, due to Rost [12] and related in detail in Chapter 8 of Liggett [10].

In the original version of the paper we did not have a "process" proof of $c \leq 2$, instead presenting an unpublished proof of Vershik - Kerov which uses Young tableaux to show $EL_n \leq 2\sqrt{n}$ for all $n \geq 1$. An anonymous referee provided a proof of $c \leq 2$ using Hammersley's process, and we thank the referee for allowing us to use the proof (section 2.5), and for other suggestions on improving the exposition, including a simplified proof of Lemma 12.

Current knowledge of asymptotics of the distribution of L_n is recorded in section 3.

2 Hammersley's process

2.1 Superadditivity and the planar Poisson process representation

The ideas in section 2.1 are due to Hammersley [8], and were one of the motivations for the development of subadditive ergodic theory. They are now a textbook application of that theory (see [6] sec 6.7).

Consider n points (x_i, t_i) in the rectangle $[0, x] \times [0, t]$ with all coordinates distinct. The set of points specifies a permutation π by:

> the point with i'th smallest t-coordinate has the $\pi(i)$'th smallest x-coordinate.

(The set of points in figure 1 specifies the permutation (1).) The length $l(\pi)$ of the longest increasing subsequence of π equals the maximal number of points on an up-right path from $(0, 0)$ to (x, t), i.e. the maximal length l of a sequence (i_j) such that

$$x_{i_1} < x_{i_2} < \ldots < x_{i_l}, \quad t_{i_1} < t_{i_2} < \ldots < t_{i_l}.$$

Figure 1 shows the path through the points corresponding to the increasing subsequence (2).

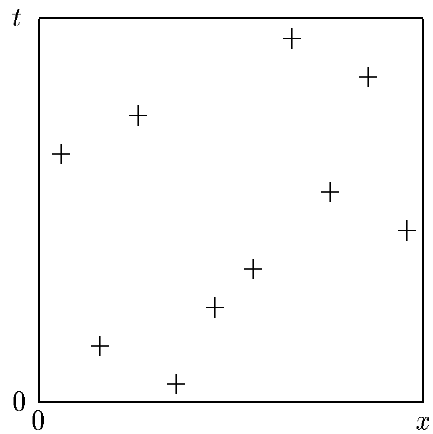

figure 1

Now take a Poisson process $\mathcal{N}$ of rate 1 in R^2 and for $x, t \geq 0$ let $\mathbf{L}^{\nearrow}(x,t)$ be the maximal number of points on an up-right path from $(0,0)$ to (x,t). The number of points in the rectangle $[0,x] \times [0,t]$, say $M(x,t)$, has Poisson(xt) distribution, and the associated random permutation of $\{1, 2, \ldots, M(x,t)\}$ is uniform. Thus

$$\mathbf{L}^{\nearrow}(x,t) \ \stackrel{d}{=}\ L_{M(x,t)}. \tag{3}$$

Define

$$g(t) = E\mathbf{L}^{\nearrow}(t,t). \tag{4}$$

By considering paths from $(0,0)$ to $(t+s, t+s)$ via (t,t) we see that g is *superadditive*:

$$g(t+s) \geq g(t) + g(s);\ s, t \geq 0. \tag{5}$$

This implies that, defining $c = \limsup g(t)/t$, we have

$$g(x)/x \to c \tag{6}$$

$$g(x) \leq cx;\ x \geq 0. \tag{7}$$

Moreover ([6] sec. 6.7) the subadditive ergodic theorem can be applied to $\mathbf{L}^{\nearrow}(t,t)$ to show

$$t^{-1}\mathbf{L}^{\nearrow}(t,t) \to c \text{ a.s.} \tag{8}$$

and simple estimates show $1.59 < c < 2.49$.

Now (3) says that L_n is almost the same as $\mathbf{L}^{\nearrow}(n^{1/2}, n^{1/2})$, and elementary dePoissonization arguments show that (6,8) imply

$$n^{-1/2} E L_n \to c, \quad n^{-1/2} L_n \stackrel{p}{\to} c.$$

2.2 Reformulation as an interacting particle process

A configuration of particles on R or R^+ may be identified with its counting process $n(\cdot)$, where $n(0)$ is arbitrary and

$$n(y) - n(x) = \text{ number of particles in } [x, y),$$

and where only finitely many particles are allowed in a finite interval. We use the phrase "particle configuration" instead of "point process" to avoid confusion with the space-time point process $\mathcal{N}$. Give the space of distributions on particle configurations the usual topology of weak convergence for point processes.

Consider, as in figure 1, a set of points (x_i, t_i) in the rectangle $[0, x^*] \times [0, t^*]$, and for (x, t) in that rectangle let $l^{\nearrow}(x, t)$ be the maximal number of points on an up-right path from $(0, 0)$ to (x, t). For each t, the function $x \to l^{\nearrow}(x, t)$ is the counting process associated with some particle configuration on $[0, x^*]$. Ordering the points (x_i, t_i) so that $0 < t_1 < t_2 < \ldots$, the particle configuration changes only at times t_i. Fix t_i and let the particles at time t_i- be at positions

$$0 < \eta_1 < \eta_2 < \ldots < \eta_m < x^*$$

and suppose $x_i \in (\eta_j, \eta_{j+1})$, where we interpret $\eta_0 = 0$ and $\eta_{m+1} = x^*$. From the definition of $l^{\nearrow}(x, t)$, the only values of x for which $l^{\nearrow}(x, t_i)$ could differ from $l^{\nearrow}(x, t_i-)$ are those with $x \in [x_i, \eta_{j+1})$, and for such x we have $l^{\nearrow}(x, t_i) - l^{\nearrow}(x, t_i-) = 1$. So at time t_i the particle configuration becomes

$$0 < \eta_1 < \ldots < \eta_j < x_i < \eta_{j+2} < \ldots$$

In words, at time t_i the particle nearest to the right of x_i is moved to position x_i, and if no such particle exists then a new particle is created at x_i.

We can apply this deterministic correspondence in the random setting of section 2.1, where $\mathbf{L}^{\nearrow}(x, t)$ is the maximal number of points of the space-time Poisson process $\mathcal{N}$ on an up-right path from $(0, 0)$ to (x, t). For fixed t the process $x \to \mathbf{L}^{\nearrow}(x, t)$ is the counting process associated with some random configuration of particles on R^+. If we fix x^* and consider only the particles in $[0, x^*]$, their time-evolution can be described as follows.

Rule 1 *At the times of a Poisson (rate x^*) process in time, a point U is chosen uniformly on $[0, x^*]$, independent of the past, and the particle nearest to the right of U is moved to U, with a new particle created at U if no such particle exists in $[0, x^*]$.*

What's really going on is that $\mathbf{L}^{\nearrow}$ defines a particle process on R^+ whose time-evolution can be described informally as

Rule 2 *For each interval $[x, x+dx]$ at time t, with probability $dx\,dt$ the nearest particle to the right of x is moved to x by time $t+dt$*

where $x, t > 0$. Rule 1 is the most elementary formalization of Rule 2.

Such processes are implicit in Hammersley [8], so a particle process evolving by Rule 2 on some interval we call *Hammersley's process*. In the language of interacting particle systems, a construction of such a process via up-right paths is a *graphical representation*. In particular, $\mathbf{L}^{\nearrow}(x,t)$ is a graphical representation of Hammersley's process on R^+ starting at time 0 with no particles. (From the viewpoint of Rule 2 it may seem surprising that one can start with no particles, but this is clear from the rigorous Rule 1.) Figure 2 shows the space-time trajectories of particles of this process. The points of $\mathcal{N}$ are at the L-shaped corners. In the sequel we will write $\mathbf{N}^+$ for Hammersley's process on R^+ started from the empty configuration, when we are considering it as an interacting particle process: of course $\mathbf{N}^+(x,t) = \mathbf{L}^{\nearrow}(x,t)$.

Of course we can start Hammersley's process on R^+ at time 0 with some arbitrary random configuration $\mathbf{N}(\cdot, 0)$ of particles. In this case, by repeating the argument above we see that the counting process $x \to \mathbf{N}(x,t)$ at time t has the graphical representation

$$\mathbf{N}(x,t) = \sup_{0 \le z \le x} (\mathbf{N}(z,0) + \mathbf{L}^{\nearrow}((z,0),(x,t))), \quad x, t \ge 0$$

where $\mathbf{L}^{\nearrow}((z,0),(x,t))$ is the maximal number of points of $\mathcal{N}$ on an up-right path from $(z,0)$ to (x,t), where $\mathcal{N}$ is taken independent of $\mathbf{N}(\cdot,0)$.

Our arguments exploit both the "interacting particle" and the "up-right path" descriptions of Hammersley's process on R^+. First note that invariance properties of the underlying space-time Poisson process $\mathcal{N}$ imply

Lemma 3 (space-time interchange property) *Write $\hat{L}(x,t) = \mathbf{L}^{\nearrow}(t,x)$. Then*

$$(\hat{L}(x,t); x, t \ge 0) \stackrel{d}{=} (\mathbf{L}^{\nearrow}(x,t); x, t \ge 0).$$

Lemma 4 (scaling property) *For fixed* $0 < \kappa < \infty$,

$$(\mathbf{L}^{\nearrow}(x,t); x,t \geq 0) \stackrel{d}{=} (\mathbf{L}^{\nearrow}(\kappa x, t/\kappa); x,t \geq 0).$$

In particular, the distribution of $\mathbf{L}^{\nearrow}(x,t)$ depends only on the product tx.

2.3 The hydrodynamic heuristic, and outline of proof

Recall that $\mathbf{N}^+(x,t) = \mathbf{L}^{\nearrow}(x,t)$ denotes Hammersley's process on R^+ started with the empty configuration. The subadditive result (8) and the scaling property (Lemma 4) imply

$$\frac{\mathbf{N}^+(x,t)}{\sqrt{tx}} \stackrel{p}{\to} c \text{ as } tx \to \infty.$$

The main result of the paper is

Theorem 5 *(a)* $c = 2$.
(b) For fixed $a > 0$*, the random particle configuration with counting process* $(\mathbf{N}^+(at+y,t) - \mathbf{N}^+(at,t), -\infty < y < \infty)$ *converges in distribution, as* $t \to \infty$*, to the Poisson process of rate* $a^{-1/2}$.

Of course (a) is already known via [14, 11], but our purpose is to give an independent "interacting particles" proof which establishes (a) along with the new result (b). The proof occupies the remainder of section 2, but let us first give a simple heuristic argument. *Suppose* the spatial process around position x at time t approximates a Poisson process of some rate $\lambda(x,t)$. Clearly

$$\frac{d}{dt} E\mathbf{N}^+(x,t) = ED_{x,t}$$

where $D_{x,t}$ is the distance from x to the nearest particle to the left of x. For a Poisson process, $ED_{x,t}$ would be 1/ (spatial rate), so

$$ED_{x,t} \approx \frac{1}{\lambda(x,t)} \approx \frac{1}{\frac{d}{dx} E\mathbf{N}^+(x,t)}.$$

In other words, $w(x,t) = E\mathbf{N}^+(x,t)$ satisfies approximately the PDE

$$\frac{dw}{dt} = \frac{1}{\frac{dw}{dx}}; \quad w(0,x) = w(t,0) = 0 \tag{9}$$

whose solution is $w(x,t) = 2\sqrt{tx}$. (Note that "2" is not an arbitrary constant: no other constant will serve.) So $c = 2$, and then $\lambda(x,t) = \sqrt{t/x}$, so that $\lambda(at,t) = a^{-1/2}$, giving (b).

"Hydrodynamics" refers to the idea that Hammersley's process should asymptotically be distributed locally like some Poisson process. This idea underlies the entire proof, but the actual proof gives separate arguments for $c \leq 2$ and $c \geq 2$, and only at the end establishes the explicit hydrodynamical limit (b). Both parts of the proof rely on the fact (section 2.4) that one can define Hammersley's process on the doubly-infinite line R, where it has Poisson processes as its stationary distributions. The upper bound (section 2.5) is then a simple comparison argument between Hammersley's process on R^+ and on R, whereas the lower bound (section 2.7) involves consideration of local subsequential weak limits of Hammersley's process on R^+ and arguing, using uniqueness properties of the stationary distribution on R, that weak limits can only be mixtures of Poisson processes. As mentioned in the introduction, many of these arguments parallel those in Rost's theorem ([12]; [10] sec. 8.5) for the non-equilibrium exclusion process.

2.4 Hammersley's process on R

By Hammersley's process on R, the doubly-infinite space line, we mean a particle configuration on R evolving according to Rule 2, where now $-\infty < x < \infty$. More precisely (cf. Rule 1) we mean any process whose restriction to each space interval $[x_0^*, x_1^*]$ evolves as
(i) there is some arbitrary set of times at which the leftmost point (if any) in the interval is removed
(ii) there is a Poisson process (rate $x_1^* - x_0^*$) of times at which a point U is picked uniformly on $[x_0^*, x_1^*]$ and the nearest particle to the right of U is moved to U, creating a new particle if necessary.
It is not quite obvious that such processes exist (e.g. if we tried to start with only a finite number of particles, they would be instantly pulled to $-\infty$), but we can use the graphical representation as a construction.

Lemma 6 *Suppose an initial configuration of particles on R satisfies*

$$\liminf_{x \to -\infty} \mathbf{N}(x,0)/x > 0 \ \ a.s. \ . \tag{10}$$

Then the process defined by

$$\mathbf{N}(x,t) = \sup_{-\infty < z \leq x} (\mathbf{N}(z,0) + \mathbf{L}^{\nearrow}((z,0),(x,t))), \ \ -\infty < x < \infty, \ t \geq 0 \tag{11}$$

evolves according to Rule 2.

Proof. The only issue is to check that $\mathbf{N}(x,t)$ is a.s. finite, and in view of (10) it is enough to check

$$\frac{\mathbf{L}^{\nearrow}((z,0),(x,t))}{-z} \to 0 \text{ a.s. as } z \to -\infty. \tag{12}$$

By scaling and (6) we have $E\mathbf{L}^{\nearrow}((z,0),(x,t)) = g(\sqrt{t(x+z)}) \sim c\sqrt{-zt}$ as $z \to -\infty$, and then by Markov's inequality the convergence in (12) holds as z runs through $(-n^4)$, and finally by monotonicity we have full convergence in (12). □

Call an initial distribution μ (time)-*invariant* for Hammersley's process on R if the distribution at each time $t > 0$ remains μ. Call μ *translation-invariant* if it is invariant under the spatial shift map. Say μ has *finite intensity* if $E|\mathbf{N}(x,0)| < \infty$ for all x. Write ν_λ for the Poisson point process of rate λ on R.

Lemma 7 *A finite intensity distribution is invariant and translation-invariant for Hammersley's process on R iff it is a mixture of the (ν_λ).*

This is analogous to the fact ([10] Theorem 8.3.9(a)) that the invariant and translation-invariant distributions for the exclusion process are precisely the mixtures of Bernoulli processes. Since the proof uses only standard ideas we shall merely outline it.

Outline proof of Lemma 7. Consider the "finite" version of Hammersley's process in which a fixed number K of particles occupy random positions on the circle of circumference C, and where the time-dynamics are as Rule 2 with "nearest particle *clockwise*" in place of "nearest particle to the right". This process is doubly-stochastic and so has uniform stationary distribution. Taking weak limits as $C \to \infty$ with $K/C \to \lambda$ shows that ν_λ is indeed invariant for Hammersley's process on R. Conversely, suppose μ is invariant and translation-invariant, and suppose also μ is spatially ergodic with rate λ. Consider the natural coupling between two versions of Hammersley's process, which at time 0 are independent with distributions μ and ν_λ, and which evolve using the same space-time Poisson process $\mathcal{N}$. Particles which become matched stay matched. The coupled processes at time t are translation-invariant, so have some spatial rate $d(t)$ of unmatched particles, where $d(t)$ is non-increasing in t. Suppose $d(t) \to d > 0$. Then by taking a subsequential weak limit of the coupled processes and using that as an initial joint distribution, we would get coupled processes with distributions

μ and ν_λ such that $d(t) = d$ for all $t \geq 0$. But this is impossible, because for a finite space interval I containing an unmatched particle from each process, there is always some possible pattern of points of $\mathcal{N}$ in $I \times [0,1]$ which would match the particles. Thus $d(t) \to 0$ and so $\mu = \nu_\lambda$. The non-ergodic case follows by conditioning on the spatial ergodic σ-field. □

Figure 3 shows space-time trajectories of particles in Hammersley's process on R run with distribution ν_1.

The next lemma is a space-time interchange property for Hammersley's process on R (cf. Lemma 3). Given a version $\mathbf{N}(x,t)$ of Hammersley's process on R, the idea is to define $\hat{\mathbf{N}}$ by interchanging space and time in the trajectory picture, i.e. by reflecting figure 3 about the 45^o diagonal. Precisely, define

$$\hat{\mathbf{N}}(x,t) = \text{ number of particles which exit the space interval } (t,\infty) \text{ during the time interval } (0,x]. \tag{13}$$

Lemma 8 *Let* $\mathbf{N}$ *be Hammersley's process on* R*, with the invariant distribution* ν_λ*, run for time* $-\infty < t < \infty$*. Define* $\hat{\mathbf{N}}$ *as at (13). Then* $\hat{\mathbf{N}}$ *is Hammersley's process on* R *with the invariant distribution* $\nu_{1/\lambda}$*.*

Proof. As at (11), given $\mathbf{N}(\cdot, t_0)$ we can define Hammersley's process on R for $t \geq t_0$ by the graphical representation

$$\mathbf{N}(x,t) = \sup_{-\infty < z \leq x} (\mathbf{N}(z,t_0) + \mathbf{L}^{\nearrow}((z,t_0),(x,t))). \tag{14}$$

Fixing x_0 and letting $t_0 \to -\infty$, it is easy to see that for $x > x_0$ we have the alternate graphical representation

$$\mathbf{N}(x,t) = \sup_{-\infty < s \leq t} (\mathbf{N}(x_0,s) + \mathbf{L}^{\nearrow}((x_0,s),(x,t))).$$

Interchanging space and time, this implies that $\hat{\mathbf{N}}$ satisfies (14) and so is a version of Hammersley's process on R. Then $\hat{\mathbf{N}}$ inherits from $\mathbf{N}$ invariance and translation-invariance, so by Lemma 7 its marginal distribution is a mixture of Poissons. To identify the process $x \to \hat{\mathbf{N}}(x,0)$ as $\nu_{1/\lambda}$ it is enough to check that the process has rate $1/\lambda$ and that the conditional rate, given an event at $x = 0$, is also $1/\lambda$. Recall $\hat{\mathbf{N}}(x,0)$ is the number of particles which cross position 0 during time $[0,x]$. The unconditional rate is clearly $E\xi_1$, where $-\xi_1$ is the position of the nearest particle to the left of 0 at time 0, and ξ_1 has exponential(λ) distribution, so the rate is $1/\lambda$ as required. A straightforward calculation shows that, conditional on a particle crossing 0 at time 0, the position $-\xi_1^*$ to which it goes is such that ξ_1^* also has exponential(λ) distribution, and so the conditional rate is also $1/\lambda$.

2.5 The upper bound $c \leq 2$

The referee observed that it is now rather simple to prove $c \leq 2$. The key idea is that Hammersley's process on R^+ can be regarded as Hammersley's process on R whose initial configuration consists of an infinite number of particles just to the left of 0. That is, if we define

$$\begin{aligned} \mathbf{N}^1(x,0) &= 0, \ x \geq 0 \\ &= -\infty, \ x < 0 \end{aligned}$$

and then use the graphical representation (11) to define $\mathbf{N}^1(x,t)$ for all $t > 0$, then the *sup* in (11) is attained at $z = 0$ and $\mathbf{N}^1(x,t) = \mathbf{L}^{\nearrow}(x,t)$ for all $x, t \geq 0$. Fix $b > 0$ and let $\mathbf{N}^2$ be the stationary version of Hammersley's process on R with distribution ν_b. Couple $\mathbf{N}^1$ and $\mathbf{N}^2$ by using the same space-time Poisson process $\mathcal{N}$ in the graphical representation. Then $\mathbf{N}^1(0,0) = \mathbf{N}^2(0,0)$ and the graphical representation (11) implies $\mathbf{N}^1(x,t) \leq \mathbf{N}^2(x,t)$ for all $x, t \geq 0$. So for all $x, t \geq 0$

$$\begin{aligned} E\mathbf{N}^1(x,t) &\leq E\mathbf{N}^2(x,t) \\ &= E\mathbf{N}^2(x,0) + E(\mathbf{N}^2(x,t) - \mathbf{N}^2(x,0)) \\ &= bx + E(\mathbf{N}^2(x,t) - \mathbf{N}^2(x,0)) \\ &\qquad \text{because } \mathbf{N}^2(\cdot,0) \text{ is a Poisson}(b) \text{ process} \\ &= bx + t/b \end{aligned}$$

because the process $\mathbf{N}^2(x,\cdot)$ counts the number of particles entering $(-\infty, x]$, and by Lemma 8 this is a Poisson$(1/b)$ process.

Minimizing over $b > 0$ gives $E\mathbf{N}^+(x,t) = E\mathbf{N}^1(x,t) \leq 2\sqrt{xt}$, and so $c \leq 2$.

2.6 A coupling construction

As a preliminary to the proof of the lower bound, we develop another coupling construction. For random particle configurations η_1, η_2, write $\eta_1 \subseteq \eta_2$ if the set of particles of η_1 is a subset of the set of particles of η_2, and write $\eta_1 \subseteq_{st} \eta_2$ if we can define $\eta_i' \stackrel{d}{=} \eta_i$ $(i = 1,2)$ such that $\eta_1' \subseteq \eta_2'$.

Lemma 9 *(a)* $\mathbf{N}^+(\cdot,t) \subseteq_{st} \mathbf{N}^+(\cdot,t+t_0)$ *for any* $t, t_0 \geq 0$.
(b) $\mathbf{N}^+(\cdot,t) \supseteq_{st} \mathbf{N}^+(\cdot + z,t)$ *for any* $t, z \geq 0$.

Proof. Given particle configurations $\mathbf{N}^1(\cdot,0), \mathbf{N}^2(\cdot,0)$ with $\mathbf{N}^1(\cdot,0) \subseteq \mathbf{N}^2(\cdot,0)$, define $\mathbf{N}^1(x,t)$ and $\mathbf{N}^2(x,t)$ to be Hammersley's process on R^+, started at time 0 with the given configurations, and run using the same space-time Poisson process $\mathcal{N}$. It is easy to verify

$$\mathbf{N}^1(\cdot,t) \subseteq \mathbf{N}^2(\cdot,t),\ t \geq 0.$$

See figure 4, where the trajectories of the two unmatched particles are indicated by $***$. (In figure 4, $\mathbf{N}^2$ initially has 2 particles in $[0,5]$ while $\mathbf{N}^1$ initially has 0 particles in $[0,5]$.) Part (a) now follows by taking $\mathbf{N}^1(\cdot,0)$ to be the empty configuration, and $\mathbf{N}^2(\cdot,0)$ to be $\mathbf{N}^+(\cdot,t_0)$.

To prove (b), fix $z \geq 0$ and let $\mathbf{N}^3$ be Hammersley's process on $[-z,\infty)$, and let $\mathbf{N}^4$ be Hammersley's process on R^+, both started with the empty configuration, and coupled by using the same space-time Poisson process $\mathcal{N}$. It is again easy to verify

$$\mathbf{N}^3(\cdot,t) \subseteq \mathbf{N}^4(\cdot,t) \text{ on } R^+,\ t \geq 0.$$

Since $\mathbf{N}^3(\cdot,t) \stackrel{d}{=} \mathbf{N}^+(\cdot+z,t)$, part (b) follows. □

Recall $g(t) \equiv E\mathbf{N}^+(t,t) \sim ct$, by (6). The next lemma uses the coupling to get a stronger, "local" convergence result. (This is somewhat analogous to the coupling proof of the renewal theorem).

Lemma 10 *$g'(z) \to c$ as $z \to \infty$.*

Proof. Lemma 9(a) implies

$$E\mathbf{N}^+(x+x_0,t) - E\mathbf{N}^+(x,t) \leq E\mathbf{N}^+(x+x_0,t+t_0) - E\mathbf{N}^+(x,t+t_0);\ t,t_0,x,x_0 \geq 0.$$

By scaling (Lemma 4) we have $E\mathbf{N}^+(x,t) = g(\sqrt{xt})$, and so the inequality above becomes

$$g\left(\sqrt{(x+x_0)t}\right) - g\left(\sqrt{xt}\right) \leq g\left(\sqrt{(x+x_0)(t+t_0)}\right) - g\left(\sqrt{x(t+t_0)}\right);\ x,x_0,t,t_0 > 0.$$

Differentiating, we see

$$\frac{d}{dx}\frac{d}{dt}\, g(\sqrt{xt}) \geq 0$$

which, after a brief calculation, implies

$$g'(z) + zg''(z) \geq 0;\ z > 0.$$

This says that $z \to zg'(z)$ is increasing. So for any fixed z_0,

$$g'(z) \geq \frac{g'(z_0)z_0}{z}; \; z > z_0. \tag{15}$$

We can now combine this with the superadditivity property. First consider an interval $[z_1, z_1 + z_2]$. By (5) $g(z_1 + z_2) - g(z_1) \geq g(z_2)$, so there exists $z_0 \in [z_1, z_1 + z_2]$ such that $g'(z_0) \geq g(z_2)/z_2$. Then by (15),

$$g'(z_1 + z_2) \geq \frac{g'(z_0)z_0}{z_1 + z_2} \geq \frac{g(z_2)z_1}{z_2(z_1 + z_2)}.$$

Putting $z_2 = \sqrt{z}$ and $z_1 = z - z_2$, gives the lower bound $\liminf_z g'(z) \geq c$. For the opposite bound, integrate (15) to get

$$g(z) - g(z_0) \geq z_0 g'(z_0) \log(z/z_0); z > z_0.$$

Use the fact (7) that $g(z) \leq cz$, set $z = (1 + \delta)z_0$ and rearrange to get

$$g'(z_0) \leq \frac{c(1 + \delta) - \frac{g(z_0)}{z_0}}{\log(1 + \delta)}.$$

So

$$\limsup_z g'(z) \leq \frac{c\delta}{\log(1 + \delta)}$$

and letting $\delta \downarrow 0$ establishes the Lemma.

Lemma 11 *For fixed* $-\infty < x, u < \infty$,

$$E\mathbf{N}^+(t + x, t + u) - E\mathbf{N}^+(t, t) \to \frac{1}{2}c(x + u) \text{ as } t \to \infty.$$

Proof.

$$E\mathbf{N}^+(t + x, t + u) - E\mathbf{N}^+(t, t) = g\left(\sqrt{(t + u)(t + x)}\right) - g(t).$$

But $\sqrt{(t + u)(t + x)} - t \to \frac{1}{2}(x + u)$, so the result follows from Lemma 10.

2.7 Proof of Theorem 5

We now combine ingredients to complete the proof of Theorem 5. For each $t > 0$ let μ_t be the distribution of $\mathbf{N}^+(\cdot + t, t)$, considered as a particle configuration on R. Lemma 11 implies that (μ_t) is tight as $t \to \infty$, and that any subsequential limit is a finite intensity process.

Lemma 12 *If $t_j \to \infty$ and $\mu_{t_j} \to \mu$ for some limit μ, then μ is translation-invariant and is an invariant distribution for Hammersley's process on R.*

Proof. Write $\mathbf{N}(\cdot, 0)$ for a particle configuration with distribution μ, and $\mathbf{N}(x, t)$ for Hammersley's process on R started with initial distribution $\mathbf{N}(\cdot, 0)$. Fix $t > 0$. By Lemma 9(a),

$$\mathbf{N}^+(t_j + \cdot, t_j + t) \supseteq_{st} \mathbf{N}^+(t_j + \cdot, t_j)$$

and so in the $j \to \infty$ limit

$$\mathbf{N}(\cdot, t) \supseteq_{st} \mathbf{N}(\cdot, 0). \tag{16}$$

And by Lemma 11, both of $E(\mathbf{N}^+(t_j + x, t_j + t) - \mathbf{N}^+(t_j, t_j + t))$ and $E(\mathbf{N}^+(t_j + x, t_j) - \mathbf{N}^+(t_j, t_j))$ have the same limit $cx/2$, for any x, and so the two processes in (16) have the same mean number of particles in finite intervals, implying

$$\mathbf{N}(\cdot, t) \stackrel{d}{=} \mathbf{N}(\cdot, 0). \tag{17}$$

Now fix $z > 0$ and repeat the argument, starting this time with Lemma 9(b), which gives

$$\mathbf{N}^+(t_j + \cdot + z, t_j + t) \subseteq_{st} \mathbf{N}^+(t_j + \cdot, t_j + t). \tag{18}$$

Taking the $j \to \infty$ limit,

$$\mathbf{N}(\cdot + z, t) \subseteq_{st} \mathbf{N}(\cdot, t).$$

Again, this becomes the equality

$$\mathbf{N}(\cdot + z, t) \stackrel{d}{=} \mathbf{N}(\cdot, t) \tag{19}$$

by using Lemma 11 to show that $E(\mathbf{N}^+(t_j + x + z, t_j + t) - \mathbf{N}^+(t_j + z, t_j + t))$ and $E(\mathbf{N}^+(t_j + x, t_j + t) - \mathbf{N}^+(t_j, t_j + t))$ have the same limit. Combining (17) and (19),

$$\mathbf{N}(\cdot + z, t) \stackrel{d}{=} \mathbf{N}(\cdot, 0)$$

for $t, z > 0$, and this also holds for $z < 0$ by reversing the inequality in (18). Thus the limit distribution μ has the required time- and translation-invariance.

Proof of Theorem 5. By Lemma 7, a subsequential limit μ in Lemma 12 must be a mixed Poisson process, i.e. a Poisson process of some random rate Λ. Next, the space-time interchange property (Lemma 3) for Hammersley's process on R^+, and the fact that μ is a subsequential limit taken along the diagonal $\{(t,t) : t \geq 0\}$ in space-time, implies that when we run Hammersley's process on R with distribution μ we get a process which is invariant under interchanging space and time. But by Lemma 8 this implies $\Lambda \stackrel{d}{=} \Lambda^{-1}$. Since Λ and Λ^{-1} are negatively correlated,

$$1 = E\Lambda\Lambda^{-1} \leq (E\Lambda)(E\Lambda^{-1}) = (E\Lambda)^2 \tag{20}$$

and so $E\Lambda \geq 1$. For (t_j) and $\mu = \text{dist } \mathbf{N}(\cdot, 0)$ as in Lemma 12, and for any $u > 0$,

$$\begin{aligned} 2uE\Lambda &= E(\mathbf{N}(2u,0) - \mathbf{N}(0,0)) \\ &\leq \liminf_j E(\mathbf{N}^+(t_j + 2u, t_j) - \mathbf{N}^+(t_j, t_j)) \qquad \text{(Fatou's lemma)} \\ &= cu \text{ by Lemma 11.} \end{aligned}$$

So

$$c \geq 2E\Lambda \tag{21}$$

and since $E\Lambda \geq 1$ we have shown $c \geq 2$. But we already proved $c \leq 2$, so we have $c = 2$. And now (21) and (20) imply $E\Lambda = 1$ and then $P(\Lambda = 1) = 1$. Thus every subsequential weak limit μ in Lemma 12 must be ν_1, and then by tightness $\mu_t \to \nu_1$. This is the "hydrodynamical limit" assertion of Theorem 5(b) in the special case $a = 1$, and the general case follows from the scaling property, Lemma 4.

3 Final remarks

When presenting this material in talks we are invariably asked what is known about the asymptotic distribution of L_n, beyond the fundamental fact that $n^{-1/2}L_n \to 2$ in L^1, so we end with some discussion of current knowledge. It turns out that Theorem 5 can be combined with "soft" arguments to give several formally new, but not unexpected, results such as (23,24). We may present the details elsewhere [1].

(a) *Concentration inequalities.* There has been recent work using martingale concentration inequalities and more general "concentration of measure" techniques to bound the spread of L_n. The latest result, due to Talagrand [13] and improving on results of Frieze [7] and Bollobás-Brightwell [2], is

Theorem 13 *Let m_n be the median of L_n. Then for $u > 0$ and all $n \geq 1$,*

$$P(L_n \geq m_n + u) \leq 2 \exp\left(-\frac{u^2}{4(m_n + u)}\right)$$

$$P(L_n \leq m_n - u) \leq 2 \exp\left(-\frac{u^2}{4m_n}\right).$$

Theorem 13 implies var $L_n = O(n^{1/2})$, and in the other direction Bollobás - Janson [3] prove that var $L_n = \Omega(n^{1/8} \log^{-3/4} n)$. It seems generally believed, by analogy with first passage percolation heuristics, that in fact var $L_n = \Theta(n^{\alpha + o(1)})$ for some $1/8 < \alpha < 1/2$.

(b) *Large deviations.* There is an asymmetry in the large deviation behavior of L_n. One anticipates

$$n^{-1/2} \log P(L_n > c' n^{1/2}) \to \alpha(c') \ \in (-\infty, 0), \quad c' > 2. \tag{22}$$

Indeed, the "Poissonized" version of this assertion follows immediately from superadditivity and Theorem 13, though we have not attempted to write out details of a dePoissonization argument. On the other hand we can show rigorously that

$$n^{-1/2} \log P(L_n < c' n^{1/2}) \to -\infty, \ c' < 2. \tag{23}$$

The argument is very similar to the corresponding result in first-passage percolation (see Kesten [9] Thm. 4.3).

(c) *Sharper asymptotics of EL_n.* We can also use "soft" arguments to show that Theorem 5 implies

$$2n^{1/2} - EL_n \to \infty. \tag{24}$$

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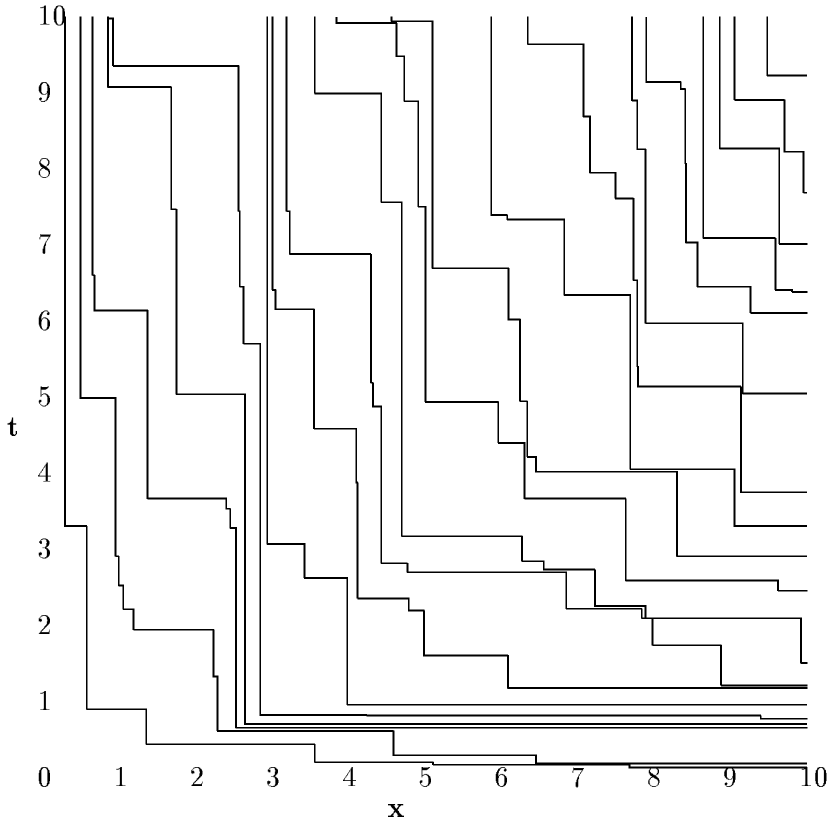

figure 2

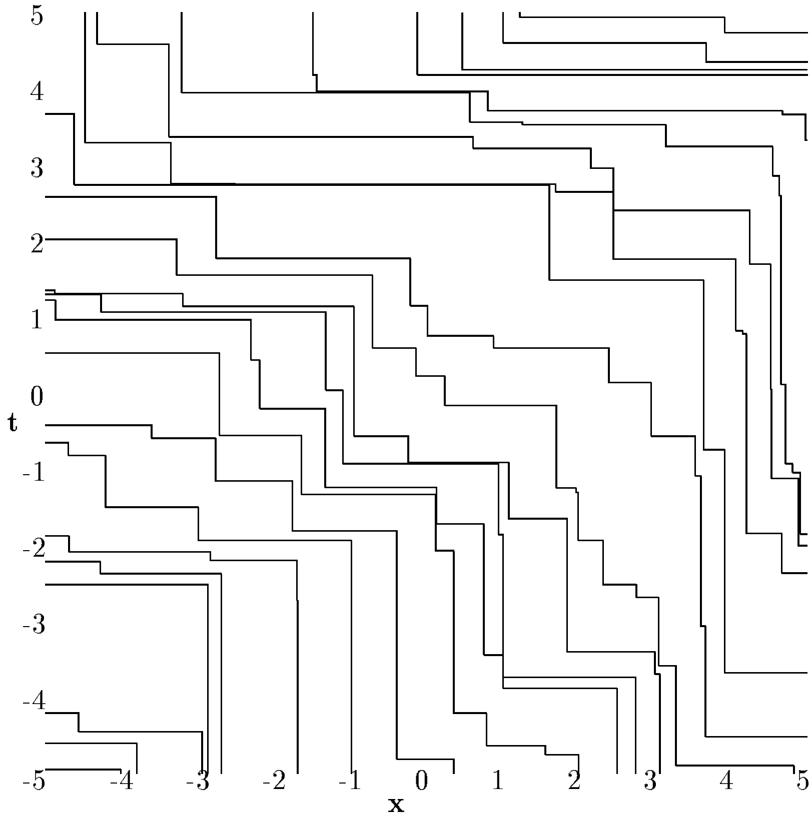

figure 3

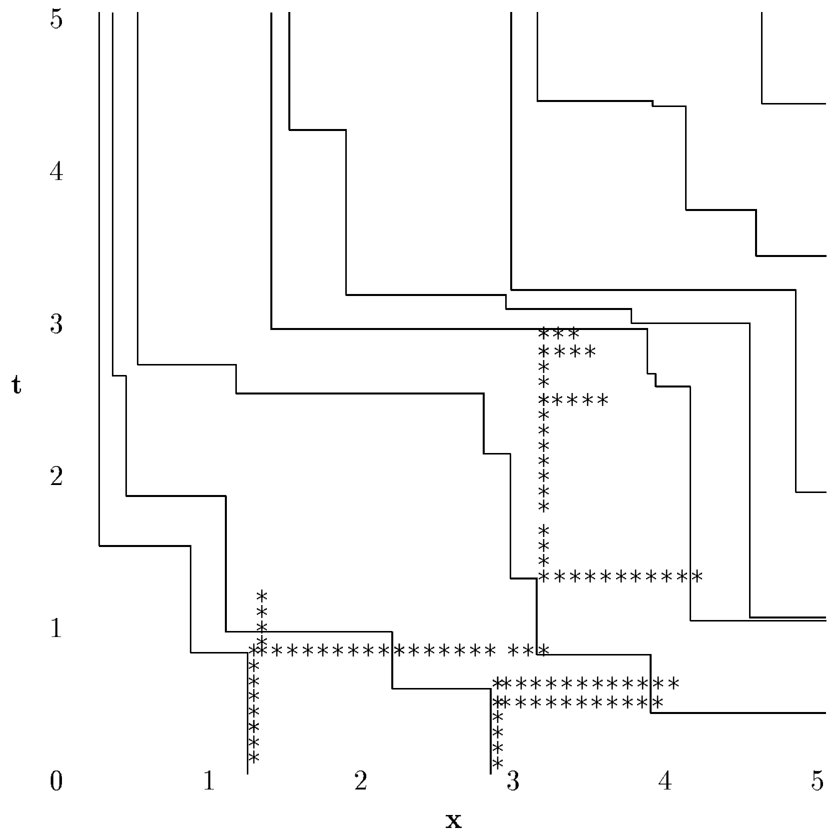

figure 4