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A guide to Brownian motion and related stochastic processes

Jim Pitman and Marc Yor

Dept. Statistics, University of California,
367 Evans Hall # 3860, Berkeley, CA 94720-3860, USA
e-mail: pitman@berkeley.edu

Abstract: This is a guide to the mathematical theory of Brownian motion and related stochastic processes, with indications of how this theory is related to other branches of mathematics, most notably the classical theory of partial differential equations associated with the Laplace and heat operators, and various generalizations thereof. As a typical reader, we have in mind a student, familiar with the basic concepts of probability based on measure theory, at the level of the graduate texts of Billingsley [43] and Durrett [106], and who wants a broader perspective on the theory of Brownian motion and related stochastic processes than can be found in these texts.

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Contents

1. Introduction

This is a guide to the mathematical theory of Brownian motion (BM) and related stochastic processes, with indications of how this theory is related to other branches of mathematics, most notably the classical theory of partial differential equations associated with the Laplace and heat operators, and various generalizations thereof.

As a typical reader, we have in mind a student familiar with the basic concepts of probability based on measure theory, at the level of the graduate texts of Billingsley [43] and Durrett [106], and who wants a broader perspective on the theory of BM and related stochastic processes than can be found in these texts. The difficulty facing such a student is that there are now too many advanced texts on BM and related processes. Depending on what aspects or applications are of interest, one can choose from any of the following texts, each of which contains excellent treatments of many facets of the theory, but none of which can be regarded as a definitive or complete treatment of the subject.

General texts on BM and related processes

[139] D. Freedman. *Brownian motion and diffusion* (1983).
[180] K. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths* (1965).
[179] K. Itô. *Lectures on stochastic processes* (1984).
[201] O. Kallenberg. *Foundations of modern probability* (2002).
[211] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus* (1988).
[206] G. Kallianpur and S. P. Gopinath. *Stochastic analysis and diffusion processes* (2014).
[221] F. B. Knight. *Essentials of Brownian motion and diffusion* (1981).
[313] P. Mörters and Y. Peres. *Brownian motion* (2010).
[370] D. Revuz and M. Yor. *Continuous martingales and Brownian motion* (1999).
[372] and [373] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales, Vols. I and II* (1994).

This list does not include more specialized research monographs on subjects closely related to BM such as stochastic analysis, stochastic differential geometry, and more general theory of Gaussian and Markov processes. Lists of such monographs classified by subject can be found in following sections.

1.1. History

The physical phenomenon of Brownian motion was discovered by Robert Brown, a 19th century scientist who observed through a microscope the random swarming motion of pollen grains in water, now understood to be due to molecular bombardment. The theory of Brownian motion was developed by Bachelier in his 1900 PhD Thesis [8], and independently by Einstein in his 1905 paper [113] which used Brownian motion to estimate Avogadro's number and the size of molecules. The modern mathematical treatment of Brownian motion (abbreviated to BM), also called the *Wiener process* is due to Wiener in 1923 [436]. Wiener proved that there exists a version of BM with continuous paths. Lévy made major contributions to the theory of Brownian paths, especially regarding the structure of their level sets, their occupation densities, and other fine features of their oscillations such as laws of the iterated logarithm. Note that BM is a Gaussian process, a Markov process, and a martingale. Hence its importance in the theory of stochastic process. It serves as a basic building block for many more complicated processes. For further history of Brownian motion and related processes we cite Meyer [307], Kahane [197], [199] and Yor [455].

1.2. Definitions

This section records the basic definition of a Brownian motion B, along with some common variations in terminology which we use for some purposes. The basic definition of B, as a random continuous function with a particular family of finite-dimensional distributions, is motivated in Section 2 by the appearance of this process as a limit in distribution of rescaled random walk paths.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $(B(t,\omega), t \geq 0, \omega \in \Omega)$ is a *Brownian motion* if

(i) For fixed each t, the random variable $B_t = B(t, \cdot)$ has Gaussian distribution with mean 0 and variance t.
(ii) The process B has *stationary independent increments*.
(iii) For each fixed $\omega \in \Omega$, the path $t \to B(t, \omega\ )$ is continuous.

The meaning of (ii) is that if $0 \leq t_1 < t_2 < \ldots < t_n$, then $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent, and the distribution of $B_{t_i} - B_{t_{i-1}}$ depends only on $t_i - t_{i-1}$. According to (i), this distribution is normal with mean 0 and variance $t_i - t_{i-1}$. We say simply that B is *continuous* to indicate that B has continuous paths as in (iii). Because of the convolution properties of normal distributions, the joint distribution of $X_{t_1}, \ldots, X_{t_n}$ are consistent for any $t_1 < \ldots < t_n$. By Kolmogorov's consistency theorem [43, Sec. 36], given such a consistent family of finite dimensional distributions, there exists a process $(\ X_t, t \geq 0\ )$ satisfying (i) and (ii), but the existence of such a process with continuous paths is not obvious. Many proofs of existence of such a process can be found in the literature. For the derivation from Kolmogorov's criterion for sample path continuity, see [370, §, Theorem (1.8)]. Freedman [139] offers a more elementary approach via the following steps:

- Step 1: Construct X_t for $t \in D\ = \{\text{dyadic rationals}\} = \{\frac{k}{2^n}\}$.
- Step 2: Show for almost all ω, $t \to X(t,\omega)$ is uniformly continuous on $D \cap [0,T]$ for any finite T.
- Step 3: For such ω, extend the definition of $X(t,\omega)$ to $t \in [0,\infty)$ by continuity.

- Step 4: Check that (i) and (ii) still hold for the process so defined.

Except where otherwise specified, a Brownian motion B is assumed to be one-dimensional, and to start at $B_0 = 0$, as in the above definition. If $\beta_t = x + B_t$ for some $x \in \mathbb{R}$ then β is a *Brownian motion started at* x. Given a Brownian motion $(B_t, t \geq 0)$ starting from 0, the process $X_t := x + \delta t + \sigma B_t$ is called a *Brownian motion started at* x *with drift parameter* δ *and variance parameter* σ^2. The notation $\mathbb{P}^x$ for probability or $\mathbb{E}^x$ for expectation may be used to indicate that B is a Brownian motion started at x rather than 0, with $\delta = 0$ and $\sigma^2 = 1$. A *d-dimensional Brownian motion* is a process

$$(B_t := (B_t^{(1)}, \dots, B_t^{(d)}); t \geq 0)$$

where the processes $B^{(i)}$ are d independent one-dimensional Brownian motions.

2. BM as a limit of random walks

Let $S_n := X_1 + \ldots + X_n$ where the X_i are independent random variables with mean 0 and variance 1, and let S_t for real t be defined by linear interpolation between integer values. Let B be a standard one-dimensional BM. It follows easily from the central limit theorem [43, Th. 27.1] that

$$(S_{nt}/\sqrt{n}, t \geq 0) \stackrel{\mathrm{d}}{\to} (B_t, t \geq 0) \text{ as } n \to \infty \tag{1}$$

in the sense of weak convergence of finite-dimensional distributions. According to *Donsker's theorem* [44, 106, 370], this convergence holds also in the path space $C[0,\infty)$ equipped with the topology of uniform convergence on compact intervals. That is to say, for every $T > 0$, and every functional f of a continuous path $x = (x_s, 0 \leq s \leq T)$ that is bounded and continuous with respect to the supremum norm on $C[0, T]$,

$$\mathbb{E}[f(S_{nt}/\sqrt{n}, 0 \leq t \leq T)] \to \mathbb{E}[f(B_t, 0 \leq t \leq T)] \text{ as } n \to \infty. \tag{2}$$

Here, for ease of notation, we suppose that the random walk $X_1, X_2, \ldots$ and the limiting Brownian motion B are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. One way to do this, which can be used to prove (1), is to use the *Skorokhod embedding* technique of constructing the $X_i = B_{T_i} - B_{T_{i-1}}$ for a suitable increasing sequence of stopping times $0 \leq T_1 \leq T_2 \cdots$ such that the $T_i - T_{i-1}$ are independent copies of T_1 with $\mathbb{E}(T_1) = \mathbb{E}(X_1^2)$. See [106, Th. 8.6.1], [44] or [370] for details, and [327] for a survey of variations of this construction. To illustrate a consequence of (1),

$$\frac{1}{\sqrt{n}} \max_{1 \leq k \leq n} S_k \stackrel{\mathrm{d}}{\to} \sup_{0 \leq t \leq 1} B_t \stackrel{d}{=} |B_t| \text{ as } n \to \infty \tag{3}$$

where the equality in distribution is due to the well known *reflection principle* for Brownian motion. This can be derived via Donsker's theorem from the corresponding principle for a simple random walk with $X_i = \pm 1$ discussed in [127], or proved directly in continuous time [43][106][139][370]. See also [222], [86] for various more refined senses in which Brownian motion may be approximated by random walks.

Many generalizations and variations of Donsker's theorem are known [44]. The assumption of independent and identically distributed X_i can be weakened in many ways: with suitable auxilliary hypotheses, the X_i can be stationary, or independent but not identically distributed, or martingale differences, or otherwise weakly dependent, with little affect on the conclusion apart from a scale factor. More interesting variations are obtained by suitable conditioning. For instance, assuming that the X_i are integer valued, let $o(\sqrt{n})$ denote any sequence of possible values of S_n with $o(\sqrt{n})/\sqrt{n} \to 0$ as $n \to \infty$. Then [108]

$$(S_{nt}/\sqrt{n}, 0 \le t \le 1 \,|\, S_n = o(\sqrt{n})) \overset{\mathrm{d}}{\to} (B_t^{\mathrm{br}}, 0 \le t \le 1) \tag{4}$$

where B^{br} is the *standard Brownian bridge*, that is, the centered Gaussian process with covariance $\mathbb{E}(B_s^{\mathrm{br}} B_t^{\mathrm{br}}) = s(1-t)$ and continuous paths which is obtained by conditioning $(B_t, 0 \le t \le 1)$ on $B_1 = 0$. Some well known descriptions of the distribution of B^{br} are [370, Ch. III, Ex (3.10)]

$$(B_t^{\mathrm{br}}, 0 \le t \le 1) \overset{d}{=} (B_t - tB_1, 0 \le t \le 1) \overset{d}{=} ((1-t)B_{t/(1-t)}, 0 \le t \le 1) \tag{5}$$

where $\overset{d}{=}$ denotes equality of distributions on the path space $C[0,1]$, and the rightmost process is defined to be 0 for $t = 1$. See Section 7.1 for further discussion of this process. Let $T_- := \inf\{n : S_n < 0\}$. Then as $n \to \infty$

$$(S_{nt}/\sqrt{n}, 0 \le t \le 1 \,|\, T_- > n) \overset{\mathrm{d}}{\to} (B_t^{\mathrm{me}}, 0 \le t \le 1) \tag{6}$$

where B^{me} is the *standard Brownian meander* [174, 53], and as $n \to \infty$ through possible values of T_-

$$(S_{nt}/\sqrt{n}, 0 \le t \le 1 \,|\, T_- = n) \overset{\mathrm{d}}{\to} (B_t^{\mathrm{ex}}, 0 \le t \le 1) \tag{7}$$

where B_t^{ex} is the *standard Brownian excursion* [200, 85]. Informally,

$$\begin{aligned} B^{\mathrm{me}} &\overset{d}{=} (B \,|\, B_t > 0 \text{ for all } 0 < t < 1) \\ B^{\mathrm{ex}} &\overset{d}{=} (B \,|\, B_t > 0 \text{ for all } 0 < t < 1, B_1 = 0) \end{aligned}$$

where $\overset{d}{=}$ denotes equality in distribution. These definitions of conditioned Brownian motions have been made rigorous in a number of ways: for instance by the method of Doob h-transforms [221, 376, 129], and as weak limits as $\varepsilon \downarrow 0$ of the distribution of B given suitable events A_ε, as in [103, 51], for instance

$$(B \,|\, \underline{B}(0,1) > -\varepsilon) \overset{\mathrm{d}}{\to} B^{\mathrm{me}} \text{ as } \varepsilon \downarrow 0 \tag{8}$$

$$(B^{\mathrm{br}} \,|\, \underline{B^{\mathrm{br}}}(0,1) > -\varepsilon) \overset{\mathrm{d}}{\to} B^{\mathrm{ex}} \text{ as } \varepsilon \downarrow 0 \tag{9}$$

where $\underline{X}(s,t)$ denotes the infimum of a process X over the interval (s,t). See Section 7.1 for further treatment of Brownian bridges, excursions and meanders.

The standard Brownian bridge arises also as a weak limit of *empirical processes*: for $U_1, U_2, \ldots$ a sequence of independent uniform $[0,1]$ variables, and

$$H_n(t) := \sqrt{n}\,(\frac{1}{n}\sum_{i=1}^{n} 1(U_i \le t) - t)$$

so that

$$\mathbb{E}[H_n(t)] = 0, \quad \mathbb{E}[(H_n(t))^2] = t(1-t)$$

it is found that

$$(H_n(t), 0 \le t \le 1) \xrightarrow{\mathrm{d}} (B_t^{\mathrm{br}}, 0 \le t \le 1)$$

in the sense of convergence of finite-dimensional distributions, and also in the sense of weak convergence in the function space $D[0,1]$ of right-continuous paths with left limits, equipped with the Skorokhod topology. See [391] for the proof and applications to empirical process theory.

Some further references related to random walk approximations are Lawler [249], Spitzer [397], Ethier and Kurtz [122], Le Gall [251].

3. BM as a Gaussian process

A *Gaussian process* with index set I is a collection of random variables $(X_t, t \in I)$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that every finite linear combination of these variables $\sum_i a_i X_{t_i}$ has a Gaussian distribution. The finite-dimensional distributions of a Gaussian process $(X_t, t \in I)$ are uniquely determined by the mean function $t \to \mathbb{E}(X_t)$, which can be arbitrary, and the covariance function $(s,t) \to \mathbb{E}(X_s X_t) - \mathbb{E}(X_s)\mathbb{E}(X_t)$, which must be symmetric and non-negative definite. A Gaussian process is called *centered* if $\mathbb{E}(X_t) \equiv 0$. Immediately from the definition of Brownian motion, there is the following characterization: a real valued process $(B_t, t \ge 0)$ is a Brownian motion starting from 0 iff

(a) (B_t) is a centered Gaussian process with covariance function

$$\mathbb{E}[B_s B_t] = s \wedge t \text{ for all } s,t \ge 0; \tag{10}$$

(b) with probability one, $t \to B_t$ is continuous.

Note that for a centered process B, formula (10) is equivalent to

$$B_0 = 0 \text{ and } \mathbb{E}[(B_t - B_s)^2] = |t-s|. \tag{11}$$

The existence of Brownian motion can be deduced from Kolmogorov's general criterion [372, Theorem (25.2)] for existence of a continuous version of a stochastic process. Specialized to a centered Gaussian process $(X_t, t \in \mathbb{R}^n)$, this shows that a sufficient condition for existence of a continuous version is that $\mathbb{E}(X_s X_t)$ should be locally Hölder continuous [372, Corollary (25.6)].

3.1. Elementary transformations

This characterization of BM as a Gaussian process is often useful in checking that a process is a Brownian motion, as in the the following transformations of a Brownian motion $(B_t, t \geq 0)$ starting from 0.

Brownian scaling For each fixed $c > 0$,

$$(c^{-1/2} B_{ct}, t \geq 0) \stackrel{d}{=} (B_t, t \geq 0). \tag{12}$$

Time shift For each fixed $T > 0$,

$$(B_{T+t} - B_T, t \geq 0) \stackrel{d}{=} (B_T, t \geq 0). \tag{13}$$

and the shifted process $(B_{T+t} - B_t, t \geq 0)$ is independent of $(B_u, 0 \leq u \leq T)$. This is a form of the *Markov property* of Brownian motion, discussed further in the next section.

Time reversal for each fixed $T > 0$

$$(B_{T-t} - B_T, 0 \leq t \leq T) \stackrel{d}{=} (B_t, 0 \leq t \leq T).$$

Time inversion

$$(tB_{1/t}, t > 0) \stackrel{d}{=} (B_t, t > 0). \tag{14}$$

3.2. Quadratic variation

Consider a subdivision of $[0, t]$ say

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = t$$

with mesh

$$\max_i (t_{n,i+1} - t_{n,i}) \to 0 \text{ as } n \to \infty.$$

Then

$$\sum_i (B_{t_{n,i+1}} - B_{t_{n,i}})^2 \to t \text{ in } L^2 \tag{15}$$

with convergence almost surely if the partitions are nested [201, Th. 13.9]. An immediate consequence is that the Brownian path has unbounded variation on every interval almost surely. This means that stochastic integrals such as $\int_0^\infty f(t) dB_t$ cannot be defined in a naive way as Lebesgue-Stieltjes integrals.

3.3. Paley-Wiener integrals

It follows immediately from (11) that

$$\mathbb{E}\left[\left(\sum_{i=1}^n a_i (B_{t_{i+1}} - B_{t_i})\right)^2\right] = \sum_{i=1}^n a_i^2 (t_{i+1} - t_i) \tag{16}$$

and hence that the *Gaussian space* generated by B is precisely the collection of Paley-Wiener integrals

$$\left\{ \int_0^\infty f(t) dB_t \text{ with } f \in L^2(\mathbb{R}_+, dt) \right\} \tag{17}$$

where the stochastic integral is by definition $\sum_{i=1}^n a_i (B_{t_{i+1}} - B_{t_i})$ for $f(t) = \sum_{i=1}^n a_i 1(t_i < t \leq t_{i+1})$, and the definition is extended to $f \in L^2(\mathbb{R}_+, dt)$ by linearity and isometry, along with the general formula

$$\mathbb{E}\left[\left(\int_0^\infty f(t) dB_t \right)^2 \right] = \int_0^\infty f^2(t) dt. \tag{18}$$

See [410, §5.1.1]for background and further discussion. The identity (18) gives meaning to the intuitive idea of dB_t as *white noise*, whose intensity is the Lebesgue measure dt. Then B_t may be understood as the total noise in $[0, t]$. This identity also suggests the well known construction of BM from a sequence of independent standard Gaussian variables (G_n) as

$$B_t = \sum_{n=0}^\infty (f_n, 1_{[0,t]}) G_n \tag{19}$$

where (f_n) is any orthonormal basis of $L^2(\mathbb{R}_+, du)$, and

$$(f_n, 1_{[0,t]}) = \int_0^t f_n(u) du$$

is the inner product of f_n and the indicator $1_{[0,t]}$ in the L^2 space. Many authors have their preferred basis: Lévy [270], [271], Knight [223], Ciesielski [81].

Note also that once one BM B has been defined on some probability space, e.g. via (19), then many other Brownian motions $B^{(k)}$ can be defined on the same probability space via

$$B_t^{(k)} = \int_0^\infty k(t, u) dB_u$$

for a bounded kernel k such that $\int_0^\infty k^2(t, u) du < \infty$ and

$$\int_0^\infty (k(t, u) - k(s, u))^2 du = (t - s) \qquad \text{for } 0 < s < t.$$

Such a setup is called a non-canonical representation of Brownian motion. Many such representations were studied by Lévy; See also Hida and Hitsuda [169], Jeulin and Yor [192].

3.4. Brownian bridges

It is useful to define, as explicitly as possible, a family of *Brownian bridges* $\{(B_u^{x,y,T}, 0 \le u \le T), x, y \in \mathbb{R}\}$ distributed like a Brownian motion $(B_u, 0 \le u \le T)$ conditioned on $B_0 = x$ and $B_T = y$. To see how to do this, assume first that $x = 0$, and write

$$B_u = \left(B_u - \frac{u}{T}B_T\right) + \frac{u}{T}B_T$$

Observe that each of the random variables $B_u - (u/T)B_T$ is orthogonal to B_T, and hence the process $(B_u - (u/T)B_T, 0 \le u \le T)$ is independent of B_T. It follows that the desired family of bridges can be constructed from an unconditioned Brownian Motion B with $B_0 = 0$ as

$$B_u^{x,y,T} = x + B_u - (u/T)B_T + (u/T)(y-x) = x + u(y-x) + \sqrt{T}B^{\mathrm{br}}(u/T)$$

for $0 \le u \le T, x, y \in \mathbb{R}$, where B^{br} is the *standard Brownian bridge* as in (4).

3.5. Fine structure of Brownian paths

Regarding finer properties of Brownian paths, such as Lévy's modulus of continuity, Kolmogorov's test, laws of the iterated lograrithm, upper and lower functions, Hausdorff measure of various exceptional sets, see Itô and McKean [181] for an early account, and Mörters and Peres [313] for a more recent exposition.

3.6. Generalizations

We mention briefly in this section a number of Gaussian processes which generalize Brownian motion by some extension of either the covariance function or the index set.

3.6.1. Fractional BM

A natural generalization of Brownian motion is defined by a centered Gaussian process with (11) generalized to

$$\mathbb{E}\left[(B_t^{(H)} - B_s^{(H)})^2\right] = |t-s|^{2H}$$

where H is called the *Hurst parameter*. This construction is possible only for $H \in (0,1]$, when such a *fractional Brownian motion* $(B_t^{(H)}, t \ge 0)$ can be constructed from a standard Brownian motion B as

$$B_t^{(H)} = C_\alpha \int_0^\infty (u^\alpha - (u-t)_+^\alpha) dB_u$$

for $\alpha = H - 1/2$ and some universal constant C_α. Early work on fractional Brownian motion was done by Kolmogorov [225], Hurst [173], Mandelbrot and Van Ness [287]. See also [414] for a historical review. It is known that fractional BM is not a semimartingale except for $H = 1/2$ or $H = 1$. See [35] for an introduction to white-noise theory and Malliavin calculus for fractional BM. Other recent texts on fractional BM are [325] [310] [34]. See also the survey article [279] on fractional Gaussian fields.

3.6.2. Lévy's BM

This is the centered Gaussian process Y indexed by $\mathbb{R}^\delta$ such that

$$Y_0 = 0 \text{ and } \mathbb{E}(Y_x - Y_y)^2 = |x - y|.$$

McKean [298] established the remarkable fact that this process has a spatial Markov property in odd dimensions, but not in even dimensions. See also [71], [80], [229].

3.6.3. Brownian sheets

Instead of a white noise governed by Lebesgue measure on the line, consider a white noise in the positive orthant of $\mathbb{R}^N$ for some $N = 1, 2, \ldots$, with intensity Lebesgue measure. Then for $t_1, \ldots, t_N \geq 0$ the noise in the box $[0, t_1] \times \cdots \times [0, t_N]$ is a centered Gaussian variable with variance $\prod_{i=1}^N t_i$, say $X_{t_1,\ldots,t_d}$, and the covariance function is

$$\mathbb{E}(X_{s_1,\ldots,s_N} X_{t_1,\ldots,t_N}) = \prod_{i=1}^{N} (s_i \wedge t_i)$$

This N parameter process is known as the *standard Brownian sheet*. See Khoshnevisan [218] for a general treatment of multiparameter processes, and Walsh [427] for an introduction to the related area of stochastic partial differential equations.

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4. BM as a Markov process

4.1. Markov processes and their semigroups

A *Markov process* X with measurable state space $(E, \mathcal{E})$ is an assembly of mathematical objects

$$X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_t)_{t\geq 0}, \{\mathbb{P}^x\}_{x\in E})$$

where

- $(\Omega, \mathcal{F})$ is a measurable space, often a *canonical space* of paths in E subject to appropriate regularity conditions;
- $(\mathcal{F}_t)$ is a filtration;
- $X_t : \Omega \to E$ is an $\mathcal{F}_t/\mathcal{E}$ measurable random variable, regarded as representing the state of the process at time t;
- $P_t : E \times \mathcal{E} \to [0, 1]$ is a *transition probability kernel* on $(E, \mathcal{E})$, meaning that $A \to P_t(x, A)$ is a probability distribution on $(E, \mathcal{E})$, and for each fixed $x \in E$, and $A \to P_t(x, A)$ is $\mathcal{E}$ measurable;
- the P_t satisfy the *Chapman-Kolmogorov equation*

$$P_{s+t}(x, A) = \int_E P_s(x, dy) P_t(y, A) \qquad (x \in E, A \in \mathcal{E}) \tag{20}$$

- under the probability law $\mathbb{P}^x$ on $(\Omega, \mathcal{F})$, the process (X_t) is Markovian relative to $(\mathcal{F}_t)$ with transition probability kernels (P_t), meaning that

$$\mathbb{E}^x[f(X_{s+t}) \,|\, \mathcal{F}_s] = (P_t f)(X_s) \qquad \mathbb{P}^x \text{ a.s.} \tag{21}$$

  where the left side is the $\mathbb{P}^x$ conditional expectation of $f(X_{s+t})$ given $\mathcal{F}_s$, with f an arbitrary bounded or non-negative $\mathcal{E}$ measurable function, and on the right side P_t is regarded as an operator on such f according to the formula

$$(P_t f)(x) = \int_E P_t(x, dy) f(y). \tag{22}$$

All Markov processes discussed here will be such that $P_0(x, A) = 1(x \in A)$, so $\mathbb{P}^x(X_0 = x) = 1$, though this condition is relaxed in the theory of Ray processes [372]. See [372, 370, 382] for background and treatment of Markov processes at various levels of generality.

The conditioning formula (21) implies the following description of the finite-dimensional distributions of X: for $0 \le t_1 < t_2 \cdots < t_n$:

$$\mathbb{P}^x(X_{t_i} \in dx_i, 1 \le i \le n) = P_{t_1}(x, dx_1) P_{t_2 - t_1}(x_1, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n)$$

meaning that for all bounded or non-negative product measurable functions f

$$\mathbb{E}^x[f(X_{t_1}, \dots, X_{t_n})] = \int_E \cdots \int_E f(x_1, \dots, x_n) P_{t_1}(x, dx_1) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n)$$

where the integration is done first with respect to x_n for fixed $x_1, \dots, x_{n-1}$, then with respect to x_{n-1}, for fixed $x_1, \dots, x_{n-2}$, and so on. Commonly, the transition kernels P_t are specified by *transition probability densities* $p_t(x, y)$ relative to some reference measure $m(dy)$ on $(E, \mathcal{E})$, meaning that $P_t(x, dy) = p_t(x, y) m(dy)$. In terms of such densities, the Chapman-Kolmogorov equation becomes

$$p_{s+t}(x, z) = \int_E m(dy) p_s(x, y) p_t(y, z) \tag{23}$$

which in a regular situation is satisfied for all $s, t \ge 0$ and all $x, z \in E$. In terms of the operators P_t defined on bounded or non-negative measurable functions on $(E, \mathcal{E})$ via (22), the Chapman-Kolmogorov equations correspond to the *semigroup property*

$$P_{s+t} = P_s \circ P_t \qquad s, t \ge 0. \tag{24}$$

The general analytic theory of semigroups, in particular the Hille-Yosida theorem [372, Theorem III (5.1)] is therefore available as a tool to study Markov processes.

Let $\mathbb{P}^0$ be Wiener measure i.e. the distribution on $\Omega := C[0, \infty)$ of a standard Brownian motion starting from 0, where Ω is equipped with the sigma-field $\mathcal{F}$ generated by the coordinate process $(X_t, t \ge 0)$ defined by $X_t(\omega) = \omega(t)$. Then standard Brownian is realized under $\mathbb{P}^0$ as $B_t = X_t$. Let $\mathbb{P}^x$ be the $\mathbb{P}^0$ distribution of $(x + B_t, t \ge 0)$ on Ω. Then a Markov process with state space $E = \mathbb{R}$, and $\mathcal{E}$ the Borel sigma-field, is obtained from this canonical setup with the *Brownian transition probability density function* relative to Lebesgue measure on $\mathbb{R}$

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \tag{25}$$

which is read from the Gaussian density of Brownian increments. The Chapman-Kolmogorov identity (23) then reflects how the sum of independent Gaussian increments is Gaussian.

This construction of Brownian motion as a Markov process generalizes straightforwardly to $\mathbb{R}^d$ instead of $\mathbb{R}$, allowing a Gaussian distribution μ_t with mean bt and covariance matrix $\sigma\sigma^T$ for some $d \times d$ matrix σ with transpose σ^T. The semigroup is then specified by

$$P_t f(x) = \int_{\mathbb{R}^d} f(x+y)\mu_t(dy) \tag{26}$$

for all bounded or non-negative Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. The corresponding Markovian laws $\mathbb{P}^x$ on the space of continuous paths in $\mathbb{R}^d$ can be defined by letting $\mathbb{P}^x$ be the law of the process $(x + \sigma B_t + bt, t \geq 0)$ where B is a *standard Brownian motion in* $\mathbb{R}^d$, that is a process whose coordinates are d independent standard one-dimensional Brownian motions. According to a famous result of Lévy [372, Theorem (28.12)], this construction yields the most general Markov process X with state space $\mathbb{R}^d$, continuous paths, and transition operators of the spatially homogenous form (26) corresponding to stationary independent increments. More general spatially homogeneous Markov processes X with state space $\mathbb{R}^d$, realized with paths which are right continuous with left limits, are known as *Lévy processes*. These correspond via (26) to convolution semigroups of probability measures $(\mu_t, t \geq 0)$ generated by an infinitely divisible distribution μ_1 on $\mathbb{R}^d$. See [372, §28], [24], [378] for treatments of Lévy processes.

Another important generalization of Brownian motion is obtained by considering Markov processes where the spatial homogeneity assumption (26) is relaxed, but continuity of paths is retained. Such Markov processes, subject to some further regularity conditions which vary from one authority to another, are called *diffusion processes*. The state space can now be $\mathbb{R}^d$, or a suitable subset of $\mathbb{R}^d$, or a manifold. The notion of infinitesimal generator of a Markovian semigroup, discussed further in Section 4.3 is essential for the development of this theory.

4.2. The strong Markov property

If B is an $(\mathcal{F}_t)$ Brownian motion then for each fixed time T the process

$$(B_{T+s} - B_T, s \geq 0) \tag{27}$$

is a Brownian motion independent of $\mathcal{F}_T$. According to the *strong Markov property* of Brownian motion, this is true also for all $(\mathcal{F}_t)$ *stopping times* T, that is random times $T : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that $(T \leq t) \in \mathcal{F}_t$ for all $t \geq 0$. The sigma-field $\mathcal{F}_T$ of events determined by time T is defined as:

$$\mathcal{F}_T := \{A \in \mathcal{F} : A \cap (T \leq t) \in \mathcal{F}_t\}$$

and the process (27) is considered only conditionally on the event $T < \infty$. See [106], [43] for the proof and numerous applications. More generally, a Markov process X with filtration $(\mathcal{F}_t)$ is said to have the strong Markov property if for all $(\mathcal{F}_t)$ stopping times T, conditionally given $\mathcal{F}_T$ on $(T < \infty)$ with $X_T = x$ the process $(X_{T+s}, s \geq 0)$ is distributed like $(X_s, s \geq 0)$ given $X_0 = x$. It is known [372, Th. III (9.4)] that the strong Markov property holds for Lévy processes, and more generally for the class of *Feller-Dynkin processes* X defined by the following regularity conditions: the state space E is locally compact with countable base, $\mathcal{E}$ is the Borel sigma field on E, and the transition probability operators P_t act in a strongly continuous way on the Banach space $C_0(E)$ of bounded continuous functions on E which vanish at infinity [201, Ch. 19], [370, III.2], [372, Def. (6.5)].

4.3. Generators

The *generator* $\mathcal{G}$ of a Markovian semigroup $(P_t)_{t\geq 0}$ is the operator defined as

$$\mathcal{G}f := \lim_{t\downarrow 0} \frac{P_t f - f}{t} \tag{28}$$

for suitable real-valued functions f, meaning that the limit exists in some sense, e.g. the sense of convergence of functions in a suitable Banach space, such as the space $C_0(E)$ involved in the definition of Feller-Dynkin processes. Then f is said to belong to the *domain of the generator*. From (28) it follows that for f in the domain of the generator

$$\frac{d}{dt} P_t f = \mathcal{G} P_t f = P_t \mathcal{G} f \tag{29}$$

and hence

$$P_t f - f = \int_0^t \mathcal{G} P_s f \, ds = \int_0^t P_s \mathcal{G} f \, ds \tag{30}$$

in the sense of strong differentiation and Riemann integration in a Banach space. In particular, if (P_t) is the semigroup of Brownian motion, then it is easily verified [372, p. 6] that for $f \in C_0(\mathbb{R})$ with two bounded continuous derivatives, the generator of the standard Brownian semigroup is given by

$$\mathcal{G}f = \tfrac{1}{2} f'' = \frac{1}{2}\frac{d^2 f}{dx^2} \tag{31}$$

In this case, the first equality in (29) reduces to *Kolmogorov's backward equation* for the Brownian transition density:

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2}\frac{\partial^2}{\partial x^2} p_t(x,y).$$

Similarly, the second equality in (29) yields *Kolmogorov's forward equation* for the Brownian transition density:

$$\frac{\partial}{\partial t} p_t(x,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2} p_t(x,y).$$

This partial differential equation is also known as the *heat equation*, due to its physical interpretation in terms of heat flow [98]. Thus for each fixed x the Brownian transition density function $p_t(x,y)$ is identified as the fundamental solution of the heat equation with a pole at x as $t \downarrow 0$. If we consider instead of standard Brownian motion B a Brownian motion with drift b and diffusion coefficient σ, we find instead of (31) that the generator acts on smooth functions of x as

$$\mathcal{G} = b\frac{d}{dx} + \tfrac{1}{2}\sigma^2 \frac{d^2}{dx^2}. \tag{32}$$

This suggests that given some space-dependent drift and variance coefficients $b(x)$ and $\sigma^2(x)$ subject to suitable regularity conditions, a Markov process which behaves when started near x like a Brownian motion with drift $b(x)$ and variance $\sigma(x)$ should have as its generator the second order differential operator

$$\mathcal{G} = b(x)\frac{d}{dx} + \tfrac{1}{2}\sigma^2(x) \frac{d^2}{dx^2} \tag{33}$$

Kolmogorov [224] showed that the semigroups of such Markov processes could be constructed by establishing the existence of suitable solutions of the Fokker-Planck-Kolmogorov equations determined by this generator. More recent approaches to the existence and uniqueness of such diffusion processes involve martingales in an essential way, as we discuss in the next section.

4.4. Transformations

The theory of Markov processes provides a number of ways of starting from one Markov process X and transforming it into another Markov process Y by some operation on the paths or law of X. The semigroup of Y can then typically be derived quite simply and explicitly from the semigroup of X. Such operations include suitable transformations of the state-space, time-changes, and killing. Starting from $X = B$ a Brownian motion in $\mathbb{R}$ or $\mathbb{R}^d$, these operations yield a rich collection of Markov processes whose properties encode some features of the underlying Brownian motion.

4.4.1. Space transformations

The simplest means of transformation of a Markov process X with state space $(E, \mathcal{E})$ is to consider the distribution of the process $(\Phi(X_t), t \geq 0)$ for a suitable measurable $\Phi : E \to E'$ for some $(E', \mathcal{E}')$. Typically, such a transformation destroys the Markov property, unless Φ respects some symmetry in the dynamics of X, as follows [372, I (14.1)]. Suppose that Φ maps E onto E', and that for each $x \in \mathcal{E}$ the $P_t(x, \cdot)$ distribution of Φ depends only on $\Phi(x)$, so that

$$\mathbb{P}^x[\Phi(X_t) \in A'] = Q_t(\Phi(x), A') \qquad (x \in E, A' \in \mathcal{E}') \tag{34}$$

for some family of Markov kernels $(Q_t, t \geq 0)$ on $(E', \mathcal{E}')$. Assuming for simplicity that X has continuous paths, that Φ is continuous, and that $\mathbb{P}^x$ governs X with semigroup (P_t) with $X_0 = x$. Then the $\mathbb{P}^x$ distribution of $(\Phi(X_t), t \geq 0)$ is that of a Markov process with semigroup (Q_t) and initial state $\Phi(x)$. Refer to Dynkin [110], Rogers-Williams [372], Rogers and Pitman [371] Glove and Mitro [154]. Let $\mathbb{Q}^y$ for $y \in \mathcal{E}'$ denote the common distribution of this process on $C([0,\infty), E')$ for all x with $\Phi(x) = y$. Then $(\mathbb{Q}^y, y \in E')$ defines the collection of laws on $C([0,\infty), E')$ of the canonical Markov process with semigroup $(Q_t, t \geq 0)$. Following is a well known example.

4.4.2. Bessel processes

Let $B_t = (B_t^{(i)}, 1 \leq i \leq \delta)$ be a δ-dimensional Brownian motion for some fixed positive integer δ. Let

$$R_t^{(\delta)} := |B_t| = \sqrt{\sum_{i=1}^{\delta} (B_t^{(i)})^2} \tag{35}$$

be the radial part of (B_t). If (B_t) is started at $x \in \mathbb{R}^\delta$, and $O_{x,y}$ is an orthogonal linear transformation of $\mathbb{R}^\delta$ which maps x to y, with $|O_{x,y}(z)| = |z|$ for all $z \in \mathbb{R}^\delta$, then $(O_{x,y}(B_t))$ is a Brownian motion starting at y whose radial part is pathwise identical to the radial part of (B_t). It follows immediately from this observation that $(R_t^{(\delta)}, t \geq 0)$ given B_0 with $|B_0| = r$ defines a Markov process, the δ-*dimensional Bessel process*, with initial state r and transition semigroup (Q_t) on $[0, \infty)$ which can be defined via (64) by integration of the Brownian transition density function over spheres in $\mathbb{R}^\delta$. That gives an explicit formula for the transition density of the Bessel semigroup in terms of Bessel functions [370, p. 446]. Since the generator of B acting on smooth functions is the half of the δ-dimensional Laplacian

$$\frac{1}{2} \sum_{i=1}^{\delta} \frac{d^2}{dx_i^2}$$

it is clear that the generator of the δ-dimensional Bessel process, acting on smooth functions with domain $[0, \infty)$ which vanish in a neighbourhood of 0, must be half of the radial part of the Laplacian, that is

$$\frac{\delta - 1}{2r} \frac{d}{dr} + \frac{1}{2} \frac{d^2}{dr^2} \tag{36}$$

In dimensions $\delta \geq 2$, it is known [370, Ch. XI] that this action of the generator uniquely determines the Bessel semigroup (Q_t), essentially because when started away from 0 the Bessel process never reaches 0 in finite time, though in two dimensions it approaches 0 arbitrarily closely at large times.

In dimension one, the process $(|B_t|, t \geq 0)$ is called *reflecting Brownian motion*. It is obvious, and consistent with the vanishing drift term in formula (36) for $\delta = 1$, that the reflecting Brownian motion started at $x > 0$ is indistinguishable from ordinary Brownian motion up until the random time T_0 that the path first hits 0. In dimension one, the expression (36) for the infinitesimal generator must be supplemented by a *boundary condition* to distinguish the reflecting motion from various other motions with the same dynamics on $(0, \infty)$, but different behaviour once they hit 0. See Harrison [165] for further treatment of reflecting Brownian motion and its applications to stochastic flow systems. See also R. Williams [445] regarding semimartingale reflecting Brownian motions in an orthant, and Harrison [166] for a broader view of Brownian networks.

The theory of Bessel processes is often simplified by consideration of the *squared Bessel process of dimension* δ which for $\delta = 1, 2, \ldots$ is simply the square of the norm of a δ-dimensional Brownian motion:

$$X_t^{(\delta)} := (R_t^{(\delta)})^2 := |B_t|^2 = \sum_{i=1}^{\delta} (B_t^{(i)})^2. \tag{37}$$

This family of processes enjoys the key *additivity property* that if $X^{(\delta)}$ and $X^{(\delta')}$ are two independent squared Bessel processes of dimensions δ and δ', started at values $x, x' \geq 0$, then $X^{(\delta)} + X^{(\delta')}$ is a squared processes of dimension $\delta + \delta'$, started at $x + x'$. As shown by Shiga and Watanabe [387], this property can be used to extend the definition of the squared Bessel process to arbitrary non-negative real values of the parameter δ. The resulting process is then a Markovian diffusion on $[0, \infty)$ with generator acting on smooth functions of $x > 0$ according to

$$\delta \frac{d}{dx} + 4x \frac{1}{2} \frac{d^2}{dx^2}$$

meaning that the process when at level x behaves like a Brownian motion with drift δ and variance parameter $4x$. See [370] and [458, §3.2] for further details. For $\delta = 0$ this is the *Feller diffusion* [126], which is the continuous state branching process obtained as a scaling limit of critical Galton-Watson branching processes in discrete time. Similarly, the squared Bessel process of dimension $\delta \geq 0$ may be interpreted as a continuous state branching process immigration rate δ. See [212], [246], [245], [276]. See also [155] for a survey and some generalizations of Bessel processes, including the Cox-Ingersoll-Ross diffusions which are of interest in mathematical finance.

4.4.3. The Ornstein-Uhlenbeck process

If $(B(t), t \geq 0)$ is a standard Brownian motion it is easily shown by Brownian scaling that the process $(e^{-r} B(e^{2r}), r \in \mathbb{R})$ is a two-sided stationary Gaussian Markov process, known as an *Ornstein-Uhlenbeck process*. See [370, Ex. III (1.13)] for further discussion.

4.5. Lévy processes

Processes with stationary independent increments, now called *Lévy processes*, were introduced by Lévy at the same time as he derived the *Lévy - Khintchine* *formula* which gives a precise representation of the characteristic functions of all infinitely divisible distributions. The interpretation of various components of the Lévy - Khintchine formula shows that Brownian motion with a general drift vector and covariance matrix is the only kind of Lévy process with continuous paths. Other Lévy processes can be constructed to have right continuous paths with left limits, using a Brownian motion for their continuous path component, and a suitably compensated integral of Poisson processes to create the jumps. Some of these Lévy processes, called *stable processes*, share with Brownian motion a generalization of the Brownian scaling property.

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5. BM as a martingale

Let $(\mathcal{F}_t, t \geq 0)$ be a filtration in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $M = (M_t)$ is called an $(\mathcal{F}_t)$ *martingale* if

1. M_t is $\mathcal{F}_t$ measurable for each $t \geq 0$.
2. $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ for all $0 \leq s \leq t$.

Implicitly here, to make sense of the conditional expectation, it is assumed that $\mathbb{E}|M_t| < \infty$. It is known that if the filtration $(\mathcal{F}_t)$ is right continuous, i.e. $\mathcal{F}_t^+ = \mathcal{F}_t$ up to null sets, then every martingale has a version which has right continuous paths (even with left limits). See [370].

If B is a standard Brownian motion relative for a filtration $(\mathcal{F}_t, t \geq 0)$, meaning that $B_{t+s} - B_t$ is independent of $\mathcal{F}_t$ with Gaussian distribution with mean 0 and variance s, for each $s, t \geq 0$, then both (B_t) and $(B_t^2 - t)$ are $(\mathcal{F}_t)$ martingales. So too is

$$M_t^{(\theta)} = \exp(\theta B_t - \theta^2 t/2)$$

for each θ real or complex, where a process with values in the complex plane, or in $\mathbb{R}^d$ for $d \geq 2$, is called a martingale if each of its one-dimensional components is a martingale.

Optional Stopping Theorem If (M_t) is a right continuous martingale relative to a right continuous filtration $(\mathcal{F}_t)$, and T is a stopping time for $(\mathcal{F}_t)$, meaning $(T \leq t) \in \mathcal{F}_t$ for each $t \geq 0$, and T is bounded, i.e., $T \leq C < \infty$ for some constant C, then

$$\mathbb{E}M_T = \mathbb{E}M_0.$$

Moreover, if an $(\mathcal{F}_t)$ adapted process M has this property for all bounded $(\mathcal{F}_t)$ stopping times T, then M is an $(\mathcal{F}_t)$ martingale. Variations or corollaries with same setup: If T is a stopping time (no bound now), then

$$\mathbb{E}(M_{T \wedge t} | \mathcal{F}_s) = M_{T \wedge s}$$

so $(M_{T \wedge t}, t \geq 0)$ is an $(\mathcal{F}_t)$-martingale. If T is a stopping time of Brownian motion B with $\mathbb{E}(T) < \infty$, then

$$\mathbb{E}(B_T) = 0 \text{ and } \mathbb{E}(B_T^2) = \mathbb{E}(T).$$

5.1. Lévy's characterization

Let

$$(B_t := (B_t^{(1)}, \dots, B_t^{(d)}); t \geq 0)$$

denote a d-dimensional process, and let

$$(\mathcal{F}_t := \sigma\{B_s, 0 \leq s \leq t\}, t \geq 0)$$

denote its filtration. It follows immediately from the property of stationary independent increments that if (B_t) is a Brownian motion, then

(i) $(B_t^{(i)})$ is an $(\mathcal{F}_t)$ martingale with continuous paths, for each i;
(ii) $(B_t^{(i)} B_t^{(j)})$ is an $(\mathcal{F}_t)$ martingale for $1 \leq i < j \leq d$;
(iii) $((B_t^{(i)})^2 - t))$ is an $(\mathcal{F}_t)$ martingale for each $1 \leq i \leq d$

It is an important result, due to Lévy, that if a d-dimensional process (B_t) has these three properties relative to the filtration $(\mathcal{F}_t)$ that it generates, then (B_t) is a Brownian motion. Note how a strong conclusion regarding the distribution of the process is deduced from what appears to be a much weaker collection of martingale properties. Note also that continuity of paths is essential: if (B_t) is any process with stationary independent increments such that $\mathbb{E}(B_1) = 0$ and $\mathbb{E}(B_1^2) = 1$, for instance $B_t := N_t - t$ where N is a Poisson process with rate 1, then both (B_t) and $(B_t^2 - t)$ are martingales.

More generally, if a process (B_t) has the above three properties relative to some filtration $(\mathcal{F}_t)$ with

$$\mathcal{F}_t \supseteq \mathcal{B}_t := \sigma\{B_s, 0 \leq s \leq t\} \qquad (t \geq 0),$$

then it can be concluded that (B_t) is an $(\mathcal{F}_t)$ *Brownian motion*, meaning that (B_t) is a Brownian motion and that for all $0 \leq s \leq t$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$.

5.2. Itô's formula

It is a key observation that if X is a Markov process with semigroup (P_t), and f and g are bounded Borel functions with $\mathcal{G}f = g$, then the equality between the first and last expressions in the Chapman-Kolmogorov equation (30) can be recast as

$$\mathbb{E}^x[f(X_T)] - f(x) = \mathbb{E}^x \int_0^T ds\, g(X_s) \qquad (T \geq 0, x \in E). \tag{38}$$

Equivalently, by application of the Markov property,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t ds\, g(X_s), t \geq 0) \text{ is a } (\mathbb{P}^x, \mathcal{F}_t) \text{ martingale} \tag{39}$$

for all $x \in E$. The optional stopping theorem applied to this martingale yields *Dynkin's formula*, [372, §10], according to which (38) holds also for $(\mathcal{F}_t)$ stopping times T with $\mathbb{E}^x(T) < \infty$. If $X = B$ is a one-dimensional Brownian motion, and $f \in C_b^2$, meaning that f, f' and f'' are all bounded and continuous, then $\mathcal{G}f = \frac{1}{2}f''$ and (39) reads

$$f(B_t) - f(B_0) = M_t^f + \tfrac{1}{2} \int_0^t f''(B_s)ds. \tag{40}$$

To identify more explicitly the martingale M_t^f appearing here, consider a subdivision of $[0, t]$ say

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,k_n} = t$$

with mesh

$$\max_i (t_{n,i+1} - t_{n,i}) \to 0 \text{ as } n \to \infty.$$

By a second order Taylor expansion,

$$f(B_t) - f(B_0) = \sum_i f'(B_{t_{n,i}})(B_{t_{n,i+1}} - B_{t_{n,i}}) + \tfrac{1}{2} \sum_i f''(\Theta_{n,i})(B_{t_{n,i+1}} - B_{t_{n,i}})^2 \tag{41}$$

for some $\Theta_{n,i}$ between $B_{t_{n,i}}$ and $B_{t_{n,i+1}}$. Since $\Theta_{n,i}$ is bounded and $(B_{t_{n,i+1}} - B_{t_{n,i}})^2$ has mean $t_{n,i+1} - t_{n,i}$ and variance a constant times $(t_{n,i+1} - t_{n,i})^2$, it is easily verified that as $n \to \infty$ there is the following easy extension of the fact (15) that the quadratic variation of B on $[0, t]$ equals t:

$$\sum_i f''(\Theta_{n,i})(B_{t_{n,i+1}} - B_{t_{n,i}})^2 - \sum_i f''(\Theta_{n,i})(t_{n,i+1} - t_{n,i}) \to 0 \text{ in } L^2 \tag{42}$$

while the second sum in (42) is a Riemann sum which approximates $\int_0^t f''(B_s)ds$ almost surely. Consequently, the first sum must converge to the same limit in L^2, and we learn from (41) that the martingale M_t^f in (40) is

$$M_t^f = \int_0^t f'(B_s)dB_s := \lim_{n\to\infty} \sum_i f'(B_{t_{n,i}})(B_{t_{n,i+1}} - B_{t_{n,i}}) \tag{43}$$

where the limit exists in the sense of convergence in probability. Thus we obtain a first version of *Itô's formula*: for f which is bounded with two bounded continuous derivatives:

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \tfrac{1}{2}\int_0^t f''(B_s)ds \tag{44}$$

where the *stochastic integral* $(\int_0^t f'(B_s)dB_s, t \geq 0)$ is an $(\mathcal{F}_t)$-martingale if B is an $(\mathcal{F}_t)$-Brownian motion. Itô's formula (48), along with an accompanying theory of stochastic integration with respect to Brownian increments dB_s, has been extensively generalized to a theory of stochastic integration with respect to semi-martingales [370]. The closely related theory of *Stratonovich stochastic integrals* is obtained by defining for instance

$$\int_0^t f'(B_s) \circ dB_s := \lim_{n\to\infty} \sum_i \tfrac{1}{2}(f'(B_{t_{n,i}}) + f'(B_{t_{n,i+1}}))(B_{t_{n,i+1}} - B_{t_{n,i}}). \tag{45}$$

This construction has the advantage that it is better connected to geometric notions of integration, such as integration of a differential form along a continuous path [176][177][302]and there is the simple formula

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) \circ dB_s. \tag{46}$$

However, the important martingale property of Itô integrals is hidden by the Stratonovich construction. See [336, Chapter 3] and [410, Chapter 8] for further comparison of Itô and Stratonovich integrals.

5.3. Stochastic integration

The theory of stochastic integration defines integration of suitable random integrands $f(t, \omega)$ with respect to random "measures" $dX_t(\omega)$ derived from suitable stochastic processes (X_t). The principal definitions in this theory, of *local martingales* and *semimartingales*, are motivated by a powerful calculus, known as *stochastic* or *Itô* calculus, which allows the representation of various functionals of such processes as stochastic integrals. For instance, the formula (48) can be justified for $f(x) = x^2$ to identify the martingale $B_t^2 - t$ as a stochastic integral:

$$B_t^2 - B_0^2 - t = 2\int_0^t B_s dB_s. \tag{47}$$

Similarly, the previous derivation of formula (43) is easily extended to a Brownian motion B in $\mathbb{R}^\delta$ for $\delta = 1, 2, 3, \ldots$ to show that for $f \in C^2(\mathbb{R}^\delta)$

$$f(B_t) - f(B_0) = \int_0^t (\nabla f)(B_s) \cdot dB_s + \tfrac{1}{2}\int_0^t (\Delta f)(B_s) ds \tag{48}$$

with ∇ the gradient operator and Δ the Laplacian. Again, the stochastic integral is obtained as a limit in probability of Riemann sums, along with the Itô formula, and the stochastic integral is a martingale in t.

It is instructive to study carefully what happens in Itô's formula (48) for $\delta \geq 2$ if we take f to be a radial harmonic function with a pole at 0, say

$$f(x) = \log|x| \text{ if } \delta = 2$$

and

$$f(x) = |x|^{2-d} \text{ if } \delta \geq 3$$

these functions being solutions on $\mathbb{R}^\delta - \{0\}$ of *Laplace's equation* $\Delta f = 0$, so the last term in (48) vanishes. Provided $B_0 = x \neq 0$ the remaining stochastic integral is a well-defined almost sure limit of Riemann sums, and moreover $f(B_t)$ is integrable, and even square integrable for $\delta \geq 3$. It is tempting to jump to the conclusion that $(f(B_t), t \geq 0)$ is a martingale. But this is not the case. Indeed, it is quite easy to compute $\mathbb{E}^x f(B_t)$ in these examples, and to check for example that this function of t is strictly decreasing for $\delta \geq 3$. For $\delta = 3$, according to a relation discussed further in Section 7.1, $\mathbb{E}^x(1/|B_t|)$ equals the probability that a one-dimensional Brownian motion started at $|x|$ has not visited 0 before time t. The process $(f(B_t), t \geq 0)$ in these examples is not a martingale but rather a *local martingale*.

Let X be a real-valued process, and assume for simplicity that X has continous paths and $X_0 = x_0$ for some fixed x_0. Such a process X is called a *local martingale* relative to a filtration $(\mathcal{F}_t)$ if for each $n = 1, 2, \ldots$, the stopped process

$$(X_{t \wedge T_n(\omega)}(\omega), t \geq 0, \omega \in \Omega) \text{ is an } (\mathcal{F}_t) \text{ martingale}$$

for some sequence of stopping times T_n increasing to ∞, which can be taken without loss of generality to be $T_n := \inf\{t : |X_t| > n\}$. For any of the processes $(f(B_t), t \geq 0)$ considered above for a harmonic function f with a pole at 0, these processes stopped when they first hit $\pm n$ are martingales, by consideration of Itô's formula (48) for a C^2 function $\hat{f}$ which agrees with f where f has values in $[-n, n]$, and is modified elsewhere to be C^2 on all of $\mathbb{R}^\delta$. Consideration of these martingales obtained by stopping processes is very useful, because by application of the optional sampling theorem they immediately yield formulae for hitting probabilities of the radial part of B in $\mathbb{R}^\delta$, as discussed in [372, I.18]. A *continuous semimartingale* X is the sum of a continuous local martingale and a process with continous paths of locally bounded variation.

Given a filtration $(\mathcal{F}_t)$, a process H of the form

$$H_s(\omega) := \sum_{i=1}^{k} H_i(\omega) 1[T_i(\omega) < s \leq T_{i+1}(\omega)]$$

for an increasing sequence of stopping times T_i, and H_i an $\mathcal{F}_{T_i}$ measurable random variable, is called an *elementary predictable process.* If B is an $(\mathcal{F}_t)$ Brownian motion, and H is such an elementary predictable process, one can define

$$\int_0^t H_s dB_s := \sum_{i=1}^{k} H_i (B_{t \wedge T_{i+1}} - B_{t \wedge T_i}) \tag{49}$$

and check the identity

$$\mathbb{E}\left[\left(\int_0^t H_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^t H_s^2 ds\right] \tag{50}$$

which allows the definition (49) to be extended by completion in L^2 to any pointwise limit H of elementary predictable processes such that

$$\mathbb{E}\left[\int_0^t H_s^2 ds\right] < \infty \tag{51}$$

for each $t > 0$, and the identity (50) then holds for such a limit process H. Replacing $H(s, \omega)$ by $H(s, \omega) 1(s \leq T(\omega))$, it follows that both

$$\int_0^t H_s dB_s \text{ and } \left(\int_0^t H_s dB_s\right)^2 - \int_0^t H_s^2 ds \tag{52}$$

define $(\mathcal{F}_t)$ martingales.

A similar stochastic integral, with B replaced by an $(\mathcal{F}_t)$ martingale or even a local martingale M, is obtained hand in hand with the existence of an increasing process $\langle M \rangle$ such that

$$(M_t^2 - \langle M \rangle_t, t \geq 0) \text{ is an } (\mathcal{F}_t) \text{ martingale}$$

and the above discussion (49) -(50) -(51) -(52) generalizes straightforwardly with dM_s instead of dB_s and $d\langle M \rangle_s$ instead of ds. In particular, there is then the formula

$$M_t^2 - M_0^2 - \langle M \rangle_t = 2 \int_0^t M_s dM_s$$

of which (53) is the special case for $M = B$. See e.g. Yor [450] and Lenglart [268] for details of this approach to Itô's formula for continuous semimartingales.

Let M be a continuous $(\mathcal{F}_t)$ local martingale and A an $(\mathcal{F}_t)$ adapted continuous process of locally bounded variation. Then for suitably regular functions $f = f(m, a)$ there is the following form of Itô's formula for semimartingales:

$$f(M_t, A_t) - f(M_0, A_0) = \int_0^t f'_m(M_s, A_s) dM_s + \int_0^t f'_a(M_s, A_s) dA_s + \frac{1}{2} \int_0^t f''_{m,m}(M_s, A_s) d\langle M \rangle_s \tag{53}$$

where

$$f'_m(m, a) := \frac{\partial}{\partial m} f(m, a); \quad f'_a(m, a) := \frac{\partial}{\partial a} f(m, a); f''_{m,m}(m, a) := \frac{\partial}{\partial m} \frac{\partial}{\partial m} f(m, a).$$

Note that the first integral on the right side of (53) defines a local martingale, and that the sum of the second and third integrals is a process of locally bounded variation. It is the third integral, involving the second derivative $f_{m,m}$, which is the special feature of Itô calculus.

More generally, for a vector of d local martingales $M = (M^{(i)}, 1 \le i \le d)$, and a process A of locally bounded variation, Itô's formula reads

$$f(M_t, A_t) - f(M_0, A_0) = \int_0^t \nabla_m f(M_s, A_s) \cdot dM_s + \int_0^t \nabla_a f(M_s, A_s) \cdot dA_s$$

$$+ \frac{1}{2} \int_0^t \sum_{i,j=1}^d f''_{m_i, m_j}(M_s, A_s) d\langle M^{(i)}, M^{(j)} \rangle_s$$

where for two local martingales $M^{(i)}$ and $M^{(j)}$, their *bracket* $\langle M^{(i)}, M^{(j)} \rangle$ is the unique continuous process C with bounded variation such that $M_t^{(i)} M_t^{(j)} - C_t$ is a local martingale. See Section 12.1 for connections between Itô's formula and various second order partial differential equations.

5.4. Construction of Markov processes

5.4.1. Stochastic differential equations

One of Itô's fundamental insights was that the theory of stochastic integration could be used to construct a diffusion process X in $\mathbb{R}^N$ as the solution of a stochastic differential equation

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt \tag{54}$$

for $\sigma : \mathbb{R}^N \to \mathbb{M}^{N \times N}$ a field of matrices and $b : \mathbb{R}^N \to \mathbb{R}^N$ a vector field. More formally, the meaning of (54) with initial condition $X_0 = x \in \mathbb{R}^N$ is that

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds. \tag{55}$$

It is known that under the hypothesis that σ and b are Lipschitz, the equation (55) has a *unique strong solution*, meaning that for a given Brownian motion B and initial point x the path of X is uniquely determined almost surely for all $t \geq 0$. Moreover, this solution is obained as the limit of the classical Picard iteration procedure, the process X is adapted to the filtration $(\mathcal{F}_t)$ generated by B, and the family of laws of X, indexed by the initial point x, defines a Markov process of Feller-Dynkin type with state space $\mathbb{R}^N$. See [370] for details. For a given Brownian motion B, one can consider the dependence of the solution X_t in (55) in the initial state x, say $X_t = X_t(x)$. Under suitable regularity conditions the map $x \to X_t(x)$ defines a random diffeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$. This leads to the notion of a *Brownian flow of diffeomorphisms* as presented by Kunita [236].

For $f \in C_b^2$, Itô's formula applied to $f(X_t)$ shows that

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{G}f(X_s)ds \text{ is an } (\mathcal{F}_t) \text{ martingale,}$$

where

$$\mathcal{G}f(x) := \tfrac{1}{2}\sum_{i,j} a^{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_i b_i(x)(x)\frac{\partial f}{\partial x_i}(x) \tag{56}$$

with $a(x) := (\sigma^T\sigma)(x)$. Thus, the infinitesimal generator of X, restricted to C_b^2, is the elliptic operator $\mathcal{G}$ defined by (56). To see the probabilistic meaning of the coefficients $a^{ij}(x)$, observe that if we write $X_t = (X_t^i, 1 \leq i \leq N)$ and M_t^i instead of M_t^f for $f(x) = x_i$, so

$$M_t^i = X_t^i - X_0^i - \int_0^t b_i(X_s)ds$$

then

$$\langle M^i, M^j \rangle_t = \int_0^t a^{ij}(X_s)ds$$

and more generally

$$\langle M^f, M^g \rangle_t = \int_0^t ds\Gamma(f,g)(X_s)ds$$

where

$$\Gamma(f,g)(x) := \nabla f(x) \cdot (a(x)\nabla g(x)) = \mathcal{G}(fg)(x) - f(x)\mathcal{G}g(x) - g(x)\mathcal{G}f(x) \tag{57}$$

is the *square field operator* (*opérateur carré du champ*). In particular, $a^{ij}(x) = \Gamma(x_i, x_j)(x)$. For a Markov process X on a more general state space, Kunita [236] takes a basis of functions (u_i) with respect to which he considers an infinitesimal generator of the form

$$\mathcal{G}f = \tfrac{1}{2}\sum_{i,j}\Gamma(u_i,u_j)(x)\frac{\partial^2 f}{\partial u_i \partial u_j}(x) + \cdots$$

with $\cdots$ a sum of drift terms of first order and integral terms related to jumps. See Bouleau and Hirsch [59] for further developments.

5.4.2. One-dimensional diffusions

The Bessel processes defined as the radial parts of Brownian motion in $\mathbb{R}^\delta$ are examples of *one-dimensional diffusions*, that is to say strong Markov processes with continuous paths whose state space is a subinterval of $\mathbb{R}$. Such processes have been extensively studied by a number of approaches: through their transition densities as solutions of a suitable parabolic differential equation, by space and time changes of Brownian motion, and as solutions of stochastic differential equations. For instance, an *Ornstein-Uhlenbeck process* X may be defined by the stochastic differential equation (SDE)

$$X_0 = x; \quad dX_t = dB_t + \lambda X_t dt$$

for $x \in \mathbb{R}$ and a constant $\lambda > 0$. This SDE is taken to mean

$$X_t = x + B_t + \lambda \int_0^t X_s ds$$

which is one of the rare SDE's which can be solved explicitly:

$$X_t = e^{\lambda t} \left(x + \int_0^t e^{-\lambda s} dB_s \right).$$

The result is a Gaussian Markov process which admits a number of alternative representations. See Nelson [319] for the physical motivation and background.

Following is a list of texts on one-dimensional diffusions:

[139] D. Freedman. *Brownian motion and diffusion* (1983).
[181] K. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths* (1965).
[288] P. Mandl. *Analytical treatment of one-dimensional Markov processes* (1968).
[372] and [373] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales, Vols. 1 and 2* (1994).

See also [57] for an extensive table of Laplace transforms of functionals one-dimensional diffusions, including Brownian motion, Bessel processes and the Ornstein-Uhlenbeck process.

5.4.3. Martingale problems

Kunita [233] used (39) to define the *extended infinitesimal generator* $\mathcal{G}$ of a Markov process X by the correspondence between pairs of bounded Borel functions f and g such that (39) holds. This leads to the idea of defining the family of probability measures P^x governing a Markov process X via the *martingale problem* of finding $\{\mathbb{P}^x\}$ such that (39) holds whenever $\mathcal{G}f = g$ for some prescribed infinitesimal generator $\mathcal{G}$ such as (32). This program was carried out for diffusion processes by Stroock and Varadhan [411]. See also [372, §III.13], [370, Chapter VII]. Komatsu [226] and Stroock [402] treat the case of Markov processes with jumps.

5.4.4. Dirichlet forms

The square field operator Γ introduced in (57) leads to naturally to consideration of the *Dirichlet form*

$$\epsilon_\mu(f,g) := \int \mu(dx)\Gamma(f,g)(x)$$

where μ is an invariant measure for the Markov process. In the case of Brownian motion on $\mathbb{R}^N$, the Dirichlet form is

$$\int_d x \nabla f(x) \cdot \nabla g(x)$$

where dx is Lebesgue measure. A key point is that this operator on pairs of functions f and g, which makes sense for f and g which may not be twice differentiable, can be used to characterize BM. See the following texts for development of this idea, and the general notion of a *Dirichlet process* which can be built from such an operator.

[145] M. Fukushima, Y. Ōshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, (1994)
[125] E. Fabes, M. Fukushima, L. Gross, C. Kenig, M. Röckner, and D. W. Stroock. *Dirichlet forms*, (1993).
[144] M. Fukushima. *Dirichlet forms and Markov processes*, (1980).
[59] N. Bouleau and F. Hirsch. *Dirichlet forms and analysis on Wiener space*, (1991).
[280] Z. M. Ma, M. Röckner, and J. A. Yan, editors. *Dirichlet forms and stochastic processes.* (1995).
[281] Z. M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms* (1992).
[194] J. Jost, W. Kendall, U. Mosco, M. Röckner and K.-T. Sturm. *New directions in Dirichlet forms*, (1998).

5.5. Brownian martingales

5.5.1. Representation as stochastic integrals

A (local) martingale relative to the natural filtration $(\mathcal{B}_t, t \geq 0)$ of a d-dimensional Brownian motion

$$(B_t^{(1)}, \ldots, B_t^{(d)}); t \geq 0)$$

is called a *Brownian (local) martingale* . According to an important result of Itôand Kunita- Watanabe [237], every Brownian local martingale admits a continuous version $(M_t, t \geq 0)$ which may be written as

$$M_t = c + \int_0^t m_s \cdot dB_s \tag{58}$$

for some constant c and some $\mathbb{R}^d$-valued predictable process $(m_s, s \geq 0)$ such that $\int_0^t |m_s|^2 ds < \infty$.

In particular, every $L^2(\mathcal{B}_\infty)$ random variable Y may be represented as

$$Y = \mathbb{E}(Y) + \int_0^\infty y_s \cdot dB_s \tag{59}$$

for some $\mathbb{R}^d$-valued predictable process $(y_s, s \geq 0)$ such that

$$\mathbb{E}\left[\int_0^\infty |y_s|^2 ds\right] < \infty. \tag{60}$$

Such a representing process is unique in $L^2(\Omega \times \mathbb{R}_+, \mathbb{P}(\mathcal{B}), d\mathbb{P}\, ds)$. The Clark-Ocone formula [82] [329] gives some general expression for $(y_s, s \geq 0)$ in terms of Y. This expression plays an important role in Malliavin calculus. See references in Section 11.3.

A class of examples of particular interest arises when

$$E[Y \mid \mathcal{B}_t] = \Phi(t, \omega; B_t(\omega))$$

for suitably regular $\Phi(t, \omega, x)$. In particular, if Φ is of bounded variation in t, and sufficiently smooth in x, one of Kunita's extensions of Itô's formula gives

$$E[Y \mid \mathcal{B}_t] = E(Y) + \int_0^t \nabla_x \Phi(s, \omega; B_s(\omega)) \cdot dB_s \tag{61}$$

where ∇_x is the gradient operator with respect to x. So, with the notations (58) and (59), we get, for the L^2-martingale $M_t = E[Y \mid \mathcal{B}_t]$,

$$m_s = y_s = \nabla_x \Phi(s, \omega; B_s(\omega)) \tag{62}$$

See [389, 157] for some interesting examples of such computations.

5.5.2. *Wiener chaos decomposition*

These representation results (58) and (59) may also be regarded as corollaries of the *Wiener chaos decomposition* of $L^2(\mathcal{B}_\infty)$ as

$$L_2(\mathcal{B}_\infty) = \bigoplus_{n=0}^{\infty} C_n \tag{63}$$

where C_n is the subspace of $L^2(\mathcal{B}_\infty)$ spanned by nth order multiple integrals of the form

$$\int_0^\infty dB_{t_1}^{(i_1)} \int_0^{t_1} dB_{t_2}^{(i_2)} \cdots \int_0^{t_{n-1}} dB_{t_n}^{(i_n)} f_n(t_1, \ldots, t_n)$$

for f_n subject to

$$\int_{0 \leq t_n \leq t_{n-1} \leq \cdots \leq t_1} dt_1 \cdots dt_n\, f_n^2(t_1, \ldots, t_n) < \infty$$

and $1 \leq i_j \leq d$ for $1 \leq j \leq n$. This space C_n, consisting of iterated integrals obtained from *deterministic* functions f_n, is called the nth Wiener chaos.

To prove the martingale representation (58)-(59), it suffices to establish (59) for the random variable

$$Y = \exp\left\{\int_0^\infty f(u) \cdot dB_u - \tfrac{1}{2}\int_0^\infty |f(u)|^2 du\right\}.$$

for $f \in L^2(\mathbb{R}_+ \to \mathbb{R}^d; du)$. The formula (59) now follows from Itô's formula, with

$$y_s = f(s) \exp\left\{\int_0^s f(u) \cdot dB_u - \tfrac{1}{2}\int_0^s |f(u)|^2 du\right\}.$$

Similarly, the Wiener chaos representation (63) follows by consideration of the generating function

$$\exp(\lambda x - \tfrac{1}{2}\lambda^2 u) = \sum_{n=0}^\infty \frac{\lambda^n}{n!} H_n(x,u)$$

of the Hermite polynomials $H_n(x,u)$, using the consequence of Itô's formula and $(\partial/\partial x)H_n = H_{n-1}$ that

$$H_n\left(\int_0^t f(s) \cdot dB_s, \int_0^t |f(s)|^2 ds\right) = \int_0^t H_{n-1}\left(\int_0^s f(u) \cdot dB_u, \int_0^s |f(u)|^2 du\right) f(s) \cdot dB_s.$$

We discuss in Section 6.4 some techniques for identifying the distribution of Brownian functionals in $C_0 \bigoplus C_2$.

It is known that if $X \in \bigoplus_{k=0}^n C_k$ then there exists $\alpha > 0$ such that

$$\mathbb{E}\left[\exp(\alpha |X|^{2/n})\right] < \infty$$

and also some α_0 such that for any $\beta > \alpha_0$

$$\mathbb{E}\left[\exp(\alpha |X|^{2/n})\right] = \infty$$

assuming that $X = X_0 + \cdots + X_n$ with $X_i \in C_i$ and $X_n \neq 0$. This gives some indication of the tail behaviour of distributions of various Brownian functionals Such results may be found in the book of Ledoux and Talagrand [266]. We do not know of any exact computation of the law of a non-degenerate element of C_3, or of $C_0 \bigoplus C_1 \bigoplus C_2 \bigoplus C_3$. We regard as "degenerate" a variable such as $(\int_0^\infty f(s) dB_s)^3$, whose law can be found by simple transformation of the law of some element of $C_0 \bigoplus C_1 \bigoplus C_2$.

5.6. *Transformations of Brownian motion*

In this section, we examine how a BM $(B_t, t \geq 0)$ is affected by the following sorts of changes:

Locally absolutely continuous change of probability: The background probability law $\mathbb{P}$ is modified by some density factor D_t on the σ-field $\mathcal{F}_t$ of events determined by B up to time t, to obtain a new probability law $\mathbb{Q}$.

Enlargement of filtration: The background filtration, with respect to which B is a Brownian motion, is enlarged in some way which affects the description of B as a semimartingale.

Time change: The time parameter $t \geq 0$ is replaced by some increasing family of stopping times $(\tau_u, u \geq 0)$.

The scope of this discussion can be expanded in many ways, to include e.g. the transformation induced by a stochastic differential equation, or space-time transformations, scale/speed description of a diffusion, reflection, killing, Lévy's transformation, and so on. One effect of such transformations is that simple functionals of the transformed process are just more complex functionals of BM. This has provided motivation for the study of more and more complex functionals of BM.

5.6.1. Change of probability

The assumption is that the underlying probability $\mathbb{P}$ is replaced by $\mathbb{Q}$ defined on by

$$\mathbb{Q}\Big|_{\mathcal{F}_t} = D_t \cdot \mathbb{P}\Big|_{\mathcal{F}_t} \tag{64}$$

meaning that every non negative $\mathcal{F}_t$-measurable trandom variable X_t has $\mathbb{Q}$-expectation

$$\mathbb{E}^{\mathbb{Q}} X_t := \mathbb{E}(D_t X_t).$$

This definition is consistent as t-varies, and defines a probability distribution on the entire path space, if and only if $(D_t, t \geq 0)$ is an $(\mathcal{F}_t, \mathbb{P})$ martingale. Then, Girsanov's theorem [370, Ch. VIII] states that

$$B_t = \tilde{B}_t + \int_0^t \frac{d\langle D, B\rangle_s}{D_s}, \tag{65}$$

with $(\tilde{B}_t)$ an $((\mathcal{F}_t), \mathbb{Q})$ Brownian motion. In the first instance, $(\tilde{B}_t)$ is just identified as an $((\mathcal{F}_t), \mathbb{Q})$ local martingale. But $\langle \tilde{B} \rangle = \langle B \rangle_t = t$, and hence $(\tilde{B}_t)$ is an $((\mathcal{F}_t), \mathbb{Q})$ Brownian motion, by Lévy's theorem.

This application of Girsanov's theorem has a number of important consequences for Brownian motion. In particular, for each $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, ds)$ the law $\mathbb{Q}^{(f)}$ of the process

$$\left(B_t + \int_0^t f(s)ds, t \geq 0\right) \stackrel{d}{=} \left(\tilde{B}_t + \int_0^t f(s)ds, t \geq 0\right)$$

is locally equivalent to the law $\mathbb{P}$ of BM, with the density relation

$$\mathbb{Q}^{(f)}\Big|_{\mathcal{F}_t} = D_t^{(f)} \cdot \mathbb{P}\Big|_{\mathcal{F}_t}.$$

where the density factor is

$$D_t^{(f)} = \exp\left(\int_0^t f(s)dB_s - \tfrac{1}{2}\int_0^t f^2(s)ds\right) = 1 + \int_0^t D_s^{(f)} f(s)dB_s.$$

In other words, the Wiener measure P is quasi-invariant under translations by functions F in the *Cameron-Martin space*, that is

$$F(t) = \int_0^t f(s)ds \text{ for } f \in L^2_{\mathrm{loc}}(\mathbb{R}_+, ds).$$

As a typical application of the general Girsanov formula (65), the law $\mathbb{P}_x^\lambda$ of the Ornstein-Uhlenbeck process of Section **??** is found to satisfy

$$P_x^\lambda \Big|_{\mathcal{F}_t} = \exp\left\{\frac{\lambda}{2}(B_t^2 - x^2) - \frac{\lambda^2}{2}\int_0^t B_s^2 ds\right\} \cdot \mathbb{P}^x \Big|_{\mathcal{F}_t}$$

where the formula $\frac{1}{2}(B_t^2 - x^2) = \int_0^t B_s dB_s$ has been used.

Girsanov's formula can also be applied to study the bridge of length T defined by starting a diffusion process X started at some point x at time 0, and conditioning on arrival at y at time T. Then, more or less by definition [132], for $0 < s < T$

$$\mathbb{E}^x[F(X_u, 0 \le u \le s) \,|\, X_T = y] = \mathbb{E}^x\left[F(X_u, 0 \le u \le s)\frac{p_{T-s}(X_s, y)}{p_T(x,y)}\right] \tag{66}$$

where $p_t(x,y)$ is the transition density for the diffusion. In particular, for X a Brownian bridge of length T from $(0,x)$ to (T,y) we learn from Girsanov's formula that

$$X_s = x + \beta_s + \int_0^s du\,\frac{(y - X_u)}{(T-u)} \tag{67}$$

This discussion generalizes easily to a d-dimensional Brownian motion, and to other Markov processes. See e.g. [410, §6.2.2].

5.6.2. *Change of filtration*

Consider now the description of an $(\mathcal{F}_t)$ Brownian motion (B_t) relative to some larger filtration $(\mathcal{G}_t)$, meaning that $\mathcal{G}_t \supseteq \mathcal{F}_t$ for each t. Provided $\mathcal{G}_t$ does not import too much information relative to $\mathcal{F}_t$, the Brownian motion (B_t), and more generally every $(\mathcal{F}_t)$ martingale, will remain a $(\mathcal{G}_t)$ semimartingale, or at worst a $(\mathcal{G}_t)$ *Dirichlet process*, meaning the sum of a $(\mathcal{G}_t)$ martingale and a process with zero quadratic variation. In particular, such an enlargement of filtration allows the original Brownian motion B to be decomposed as

$$B_t = \tilde{B}_t + A_t$$

where $(\tilde{B}_t)$ is (again by Lévy's characterization) a $(\mathcal{G}_t)$ Brownian motion, and (A_t) has zero quadratic variation. As an example, if we enlarge the filtration

$(\mathcal{F}_t)$ generated by (B_t) to $\mathcal{G}_t$ generated by $\mathcal{F}_t$ and $\int_0^\infty f(s)dB_s$ for some $f \in L^2(\mathbb{R}_+, ds)$, then

$$B_t = \tilde{B}_t + \int_0^t \frac{ds\, f(s) \int_s^\infty f(u)dB_u}{\int_s^\infty f^2(u)du} \tag{68}$$

where $(\tilde{B}_t)$ is independent of the sigma-field $\mathcal{G}_0$ of events generated by $\int_0^\infty f(u)dB_u$. The best known example arises when $f(s) = 1(0 \le s \le T)$, so $\mathcal{G}_t$ is generated by $\mathcal{F}_t$ and B_T. Then (68) reduces to

$$B_t = \tilde{B}_t + \int_0^{t\wedge T} \frac{ds\,(B_T - B_s)}{(T-s)}.$$

Since $(\tilde{B}_t)$ and B_T are independent, we can condition on $B_T = y$ to deduce that the Brownian bridge (B_t^{br}) of length T from $(0,0)$ to (T,x) can be related to an unconditioned Brownian motion $\tilde{B}$ by the equation

$$B_t^{\mathrm{br}} = \tilde{B}_t + \int_0^{t\wedge T} \frac{ds\,(x - B_s^{\mathrm{br}})}{(T-s)}.$$

which can be solved explicitly to give

$$B_t^{\mathrm{br}} = (t/T)y + (T-t)\int_0^t \frac{d\tilde{B}_s}{(T-s)}$$

or again, by time-changing

$$B_t^{\mathrm{br}} = (t/T)y + (T-t)\beta_{t/(T(T-t))}$$

for another Brownian motion β. Compare with Section 3.4.

Other enlargements $(\mathcal{G}_t)$ of the original Brownian filtration $(\mathcal{F}_t)$ can be obtained by turning some particular random times L into $(\mathcal{G}_t)$ stopping times, so $\mathcal{G}_t$ is the σ-field generated by $\mathcal{F}_t$ and the random variable $L \wedge t$. If $L = \sup\{t : (t,\omega) \in A\}$ for some $(\mathcal{F}_t)$ predictable set A, then there is the decomposition

$$B_t = \tilde{B}_t + \int_0^{t\wedge L} \frac{d\langle B, Z^L\rangle_s}{Z_s^L} + \int_L^t \frac{d\langle B, 1 - Z^L\rangle_s}{1 - Z_s^L}$$

where $Z_t^L := \mathbb{P}(L > t \,|\, \mathcal{F}_t)$ and $(\tilde{B}_t)$ is a $(\mathcal{G}_t)$ Brownian motion.

The volume [191] provides many applications of the theory of enlargement of filtrations, in particular to provide explanations in terms of stochastic calculus to path decompositions of Brownian motion at last exit times and minimum times. See also [193]. The article of Jacod [185] treats the problem of *initial enlargement* from $(\mathcal{F}_t)$ to $(\mathcal{G}_t)$ with $\mathcal{G}_t$ the σ-field generated by $\mathcal{F}_t$ and Z for some random variable Z whose value is supposed to be known at time 0. See also [362, Ch. 6], [454, Ch. 12], and [84] for a recent overview. On the other hand, the theory of progressive enlargements has developed very little since 1985.

5.6.3. Change of time

If the time parameter $t \geq 0$ is replaced by some increasing and right continuous family of stopping times $(\tau_u, u \geq 0)$, then according to the general theory of semimartingales we obtain from Brownian motion B a process $(B_{\tau_u}, u \geq 0)$ which is a semimartingale relative to the filtration $(\mathcal{F}_{\tau_u}, u \geq 0)$. In particular, it is follows from the Burkholder-Davis-Gundy inequalities [370, §IV.4] that if $\mathbb{E}(\sqrt{\tau_u}) < \infty$ then $(B_{\tau_u}, u \geq 0)$ is a martingale. Monroe [312] showed that every semimartingale can be obtained, in distribution, as $(B_{\tau_u}, u \geq 0)$ for a suitable time change process (τ_u).

A beautiful application of Lévy's characterization of BM is the representation of continuous martingales as time-changed Brownian motions. Here is the precise statement. Let $(M_t, t \geq 0)$ be a d-dimensional continuous local martingale relative to some filtration $(\mathcal{F}_t)$, such that

(i) $\langle M^{(i)} \rangle_t = A_t$ for some increasing process (A_t) with $A_\infty = \infty$, and all i.
(ii) $\langle M^{(i)}, M^{(j)} \rangle_t \equiv 0$, which is to say that the product $M_t^{(i)} M_t^{(j)}$ is an $(\mathcal{F}_t)$ local martingale, for all $i \neq j$.

Let

$$\tau_t := \inf\{s : A_s > t\} \text{ and } B_t := M_{\tau_t}.$$

Then the process (B_t) is an $(\mathcal{F}_{\tau(t)})$ Brownian motion, and

$$M_u = B_{A_u} \qquad (u \geq 0). \tag{69}$$

Doeblin in 1940 discovered the instance of this result for $d = 1$ and $M_t = f(X_t) - \int_0^t (Lf)(X_s) ds$ for X a one-dimensional diffusion and f a function in the domain of the infinitesimal generator L of X. See [62, p. 20]. Dambis [89] and Dubins-Schwarz [101] gave the general result for $d = 1$, while Getoor and Sharpe [151] formulated it for $d = 2$, as discussed in the next subsection.

As a simple application of (69), we mention the following: if $(M_u, u \geq 0)$ is a non-negative local martingale, such that

$$M_0 = a \text{ and } \lim_{u \to \infty} M_u = 0$$

then

$$\mathbb{P}\left(\sup_{u \geq 0} M_u \geq x\right) = a/x \qquad (x \geq a). \tag{70}$$

To prove (70), it suffices thanks to (69) to check it for M a Brownian motion started at a and stopped at its first hitting time of 0. The conclusion (70) can also be deduced quite easily by optional sampling. See [328] for further results in this vein.

5.6.4. Knight's theorem

A more general result on time-changes is obtained by consideration of a d-dimensional continuous local martingale $(M_t, t \geq 0)$ relative to some filtration

$(\mathcal{F}_t)$, such that

$$\langle M^{(i)}, M^{(j)} \rangle_t \equiv 0$$

and each of the processes $\langle M^{(i)} \rangle_t$ grows to infinity almost surely, but these increasing processes are not necessarily identical. Then, Knight's theorem [370, Theorem V (1.9)] states that if $(B_u^{(i)})$ denotes the Brownian motion such that $M_t^{(i)} = B^{(i)}_{\langle M^{(i)} \rangle_t}$, then the Brownian motions $B^{(1)}, \dots, B^{(d)}$ are independent.

As an example, if $\Gamma^{(i)}$ for $1 \le i \le d$ are d disjoint Borel subsets of $\mathbb{R}$, each with positive Lebesgue measure, then

$$\int_0^t 1(B_s \in \Gamma^{(i)}) dB_s = B^{(i)} \left(\int_0^t 1(B_s \in \Gamma^{(i)}) ds \right)$$

for some independent Brownian motions $B^{(i)}$.

5.7. BM as a harness

Another characterization of BM is obtained by considering the conditional expectation of B_u for some $u \in [s,t]$ conditionally given the path of B_v for $v \notin (s,t)$. As a consequence of the Markov property of B and exchangeability of increments, there is the basic formula

$$\mathbb{E}[B_u \,|\, B_v, v \notin (s,t)] = \frac{t-u}{t-s} B_s + \frac{u-s}{t-s} B_t \qquad (0 \le s < u < t) \tag{71}$$

which just states that given the path of B outside of (s,t), the path of B on (s,t) is expected to follow the straight line from (s, B_s) to (t, B_t). Following Hammersley [161] a process B with this property is called a *harness*. D. Williams showed around 1980 that every harness with continuous paths parameterized by $[0,\infty)$ may be represented as $(\sigma B_s + \mu s, s \ge 0)$ for σ and μ two random variables which are measurable with respect to the germ σ-field

$$\cap_{0<s<t<\infty} \sigma(B_v, v \notin (s,t)).$$

See also Jacod-Protter [186] who showed that every integrable Lévy process is a harness. Further discussion and references can be found in [289].

5.8. Gaussian semi-martingales

In general, to show that a given adapted, right continuous process is, or is not, a semimartingale, may be quite subtle. This question for Gaussian processes was studied by Jain and Monrad [187], Stricker [400] [401] and Emery [117]. Interesting examples of Gaussian processes which are not semimartingales are the fractional Brownian motion, for all values of their Hurst parameter H except $1/2$ and 1. Many studies, including the development of adhoc stochastic integration, have been made for fractional Brownian motions. In particular, P. Cheridito [72] obtained the beautiful result that the addition of a fractional Brownian motion with Hurst parameter $H > 3/4$ and an independent Brownian motion produces a semimartingale.

5.9. Generalizations of martingale calculus

Stochastic calculus for processes with jumps: Meyer's appendix [304], Meyer's course [300], the books of Chung-Williams [75] and Protter [356]. Extension of stochastic calculus to *Dirichlet processes*, that is sums of a martingale and a process of vanishing quadratic variation: Bertoin [26]. Anticipative stochastic calculus of Skorokhod and others: some references are [330] [21] [338].

5.10. References

Martingales Most modern texts on probability and stochastic processes contain an introduction at least to discrete time martingale theory. Some further references are:

[148] A. M. Garsia. *Martingale inequalities: Seminar notes on recent progress.* (1973)
[323] J. Neveu. *Martingales à temps discret.* (1972).
[444] D. Williams. *Probability with martingales.* (1991)

Semi-Martingales Pioneering works:

[340] J. Pellaumail. *Sur l'intégrale stochastique et la décomposition de Doob-Meyer* (1973).
[241] A. U. Kussmaul. *Stochastic integration and generalized martingales* (1977).

The following papers present a definitive account of semi-martingales as "good integrators"

[41] K. Bichteler. *Stochastic integrators* (1979).
[42] K. Bichteler. *Stochastic integration and L^p-theory of semimartingales* (1981).
[94] C. Dellacherie. *Un survol de la théorie de l'intégrale stochastique* (1980).

Elementary treatments:

[105] R. Durrett. *Stochastic calculus: a practical introduction* (1996).
[399] J. M. Steele. *Stochastic calculus and financial applications.* (2001)

Stochastic integration: history Undoubtedly, the inventor of Stochastic Integration is K. Itô, although there were some predecessors: Paley and Wiener who integrated deterministic functions against Brownian motion, and Lévy who tried to develop a stochastic integration framework by randomizing the Darboux sums, etc... However, K. Itô stochastic integrals, which integrate, say, predictable processes against Brownian motion proved to provide the right level of generality to encompass a large number of applications. In particular, it led to the definition and solution of stochastic differential equations, for which in most cases, Picard's iteration procedure works, and thus, probabilists were handed a pathwise construction of many Markov processes, via Itô construction. To appreciate the scope of Itô achievement, we should compare the general class of

Markov processes obtained through his method with those which Feller obtained from Brownian motion via time and space changes of variables. Feller's method works extremely well in one dimension, but does not generalize easily to higher dimensions. Itô's construction was largely unappreciated until the publication of McKean's wonderful book [296] 25 years after Itô's original paper. In 1967 Paul-André Meyer expounded Séminaire de Probabilités I [299] the very important paper of Kunita and Watanabe [237] on the application of Stochastic integration to the study of martingales associated with Markov processes. Itô theory became better known in France in 1972, following a well attended course by J. Neveu, which was unfortunately only recorded in handwritten form. The next step was taken by Paul-André Meyer in his course [306] where he blended the Itô–Kunita–Watanabe development with the general theory of processes, to present stochastic integration with respect to general semi martingales. Jacod's Lecture Notes [184] built on the Strasbourg theory of predictable and dual predictable projections, and so forth, aiming at the description of all martingales with respect to the filtration of a given process, such as a Lévy process. A contemporary to Jacod's lecture notes is the book of D. Stroock and S. Varadhan [411], where the martingale problem associated with an infinitesimal generator is used to characterize and construct diffusion processes, thereby extending Lévy's characterization of Brownian motion.

[296] H. P. McKean, Jr. *Stochastic integrals* (1969).
[306] P.-A. Meyer. *Martingales and stochastic integrals. I.* (1972).
[301] P. A. Meyer. *Un cours sur les intégrales stochastiques* (1976).
[75] K. L. Chung and R. J. Williams. *Introduction to stochastic integration* (1990).
[304] P.-A. Meyer. *A short presentation of stochastic calculus* (1989).

6. Brownian functionals

6.1. Hitting times and extremes

For $x \in \mathbb{R}$, let $T_x := \inf\{t : t \geq 0, B_t = x\}$. Then for $a, b > 0$, by optional sampling,

$$\mathbb{P}^0(T_a < T_{-b}) = \frac{b}{a+b}$$

and hence

$$\mathbb{P}^0(T_x < \infty) = 1 \text{ for all } x \in \mathbb{R}.$$

Let

$$M_t := \max_{0 \leq s \leq t} B_s$$

and notice that $(T_x, x \geq 0)$ is the left continuous inverse for $(M_t, t \geq 0)$. Since $(M_t \geq x) = (T_x \leq t)$, if we know the distribution of M_t for all $t > 0$ then we know the distribution of T_x for all $x > 0$. Define the reflected path,

$$\hat{B}(t) = \begin{cases} B(t) & \text{if } t \leq T_x \\ x - (B(t) - x) & \text{if } t > T_x \end{cases}$$

By the Strong Markov Property and the fact that B and $-B$ are equal in distribution we can deduce the *reflection principle* that $\hat{B}$ and B are equal in distribution: $\hat{B} \stackrel{d}{=} B$. Rigorous proof of this involves some measurability issues: see e.g. Freedman [139, §1.3] Durrett [106] for details. Observe that for $x, y > 0$,

$$(M_t \geq x, B_t \leq x - y) = (\hat{B}_t \geq x + y)$$

so

$$\mathbb{P}^0(M_t \geq x, B_t \leq x - y) = \mathbb{P}^0(B_t \geq x + y)$$

Taking $y = 0$ in the previous expression we have

$$\mathbb{P}^0(M_t \geq x, B_t \leq x) = \mathbb{P}^0(B_t \geq x).$$

But $(B_t > x) \subset (M_t \geq x)$, so

$$\mathbb{P}^0(M_t \geq x, B_t > x) = \mathbb{P}^0(B_t > x) = \mathbb{P}^0(B_t \geq x)$$

by continuity of the distribution. Adding these two results we find that

$$\mathbb{P}^0(M_t \geq x) = 2\mathbb{P}^0(B_t \geq x)$$

So the distributions of M_t and $|B_t|$ are the same: $M_t \stackrel{d}{=} |B_t|$.

Now recall that $\mathbb{P}^0(M_t \geq x) = \mathbb{P}^0(T_x \leq t)$ so

$$\begin{aligned}
\mathbb{P}^0(T_x \leq t) &= \mathbb{P}^0(|B_t| \geq x) = \mathbb{P}^0(\sqrt{t}|B_1| \geq x) \\
&= \mathbb{P}^0(B_1^2 \geq \frac{x^2}{t}) = \mathbb{P}^0(\frac{x^2}{B_1^2} \leq t)
\end{aligned}$$

So $T_x \stackrel{d}{=} \frac{x^2}{B_1^2}$. As a check, this implies $T_x \stackrel{d}{=} x^2 T_1$, which is explained by Brownian scaling.

The joint distribution of the minimum, maximum and final value of B on an interval can be obtained by repeated reflections. See e.g. [46] and [40, §4.1] for related results involving the extremes of Brownian bridge and excursion. See also [57] for corresponding results up to various random times.

6.2. Occupation times and local times

For $f(x) = 1(x \in A)$ for a Borel set A the integral

$$\int_0^T f(B_s)ds \tag{72}$$

represents the amount of time that the Brownian path has spent in A up to time T, which might be either fixed or random. As f varies, this integral functional defines a random measure on the range of the path of B, the *random occupation measure* of B on $[0, T]$. A basic technique for finding the distribution of the integral functional (72) is provided by the method of Feynman-Kac, which is discussed in most textbooks on Brownian motion. Few explicit formulas are known, except in dimension one. A well known application of the Feynman-Kac formula is Kac's derivation of Lévy 's arcsine law for B a BM($\mathbb{R}$), that is for all fixed times T

$$\mathbb{P}^0\left(\frac{1}{T}\int_0^T 1(B_s > 0)ds \leq u\right) = \frac{2}{\pi}\arcsin(\sqrt{u}) \qquad (0 \leq u \leq 1). \tag{73}$$

See for instance [313] for a recent account of this approach, and Watanabe [430] for various generalizations to one-dimensional diffusion processes and random walks. Other generalizations of Lévy's arcsine law for occupation times were developed by Lamperti [247] and Barlow, Pitman and Yor [10], [354]. See also [67] [68]. See Bingham and Doney [48] and Desbois [96] regarding higher-dimensional analogues of the arc-sine law, and Desbois [95] [18] for occupation times for Brownian motion on a graph.

It was shown by Trotter [418] that almost surely the random occupation measure induced by the sample path of a one dimensional Brownian motion $B = (B_t, t \geq 0)$ admits a jointly continuous local time process $(L_t^x(B); x \in \mathbb{R}, t \geq 0)$ satisfying the *occupation density formula*

$$\int_0^t f(B_s)ds = \int_{-\infty}^{\infty} L_t^x(B)f(x)dx. \tag{74}$$

See [295, 221, 370] for proofs of this. Immediately from (74) there is the almost sure approximation

$$L_t^x = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\int_0^t 1(|B_s - x| \leq \epsilon) \tag{75}$$

which was used by Lévy to define the process $(L_t^x, t \geq 0)$ for each fixed x. Other such approximations, also due to Lévy, are

$$L_t^x = \lim_{\epsilon \to 0} \epsilon D[x, x+\epsilon, B, t] \tag{76}$$

where $D[x, x+\epsilon, B, t]$ is the number of downcrossings of the interval $[x, x+\epsilon]$ by B up to time t, and

$$L_t^x = \lim_{\epsilon \to 0} \sqrt{\frac{\pi\epsilon}{2}} N[x, \epsilon, B, t] \tag{77}$$

where $N[x, \epsilon, B, t]$ is derived from the random closed level set $\mathcal{Z}_x := \{s : B_s = x\}$ as the number component intervals of $[0, t] \setminus \mathcal{Z}_x$ whose length exceeds ϵ. See [370, Prop. XII.(2.9)]. According to Taylor and Wendel [416] and Perkins [341], the local time L_t^x is also the random Hausdorff ℓ-measure of $\mathcal{Z}_x \cap [0, t]$ for $\ell(v) = (2v|\log|\log v||)^{1/2}$.

6.2.1. *Reflecting Brownian motion*

For a one-dimensional BM B, let $\underline{B}_t := \inf_{0 \le s \le t} B_s$. Lévy showed that

$$(B - \underline{B}, -\underline{B}) \stackrel{d}{=} (|B|, L) \tag{78}$$

where $L := (L_t^0, t \ge 0)$ is the local time process of B at 0. This basic identity in law of processes leads a large number of identities in distribution between various functionals of Brownian motion. For instance if G_T is the time of the last 0 of B on $[0,T]$, and A_T is the time of the last minimum of B on $[0,T]$, (or the time of the last maximum), then $G_T \stackrel{d}{=} A_T$. The distribution of G_T/T and A_T/T is the same for all fixed times T, by Brownian scaling, and given by Lévy's *arcsine law* displayed later in (73). For further applications of Lévy's identity see [100].

6.2.2. *The Ray-Knight theorems*

This subsection is an abbreviated form the account of the Ray-Knight theorems in [349, §8]. Throughout this section let R denote a reflecting Brownian motion on $[0,\infty)$, which according to Lévy's theorem (78) may be constructed from a standard Brownian motion B either as $R = |B|$, or as $R = B - \underline{B}$. Note that if $R = |B|$ then for $v \ge 0$ the occupation density of R at level v up to time t is

$$L_t^v(R) = L_t^v(B) + L_t^{-v}(B) \tag{79}$$

and in particular $L_t^0(R) = 2L_t^0(B)$. For $\ell \ge 0$ let

$$\tau_\ell := \inf\{t : L_t^0(R) > \ell\} = \inf\{t : L_t^0(B) > \ell/2\}. \tag{80}$$

For $0 \le v < w$ let

$$D(v,w,t) := \text{number of downcrossings of } [v,w] \text{ by } R \text{ before } t$$

Then there is the following basic description of the process counting downcrossings of intervals up to an inverse local time [321]. See also [426] for more about Brownian downcrossings and their relation to the Ray-Knight theorems.

The process

$$(D(v, v+\varepsilon, \tau_\ell), v \ge 0)$$

is a time-homogeneous Markovian birth and death process on $\{0,1,2\ldots\}$, with state 0 absorbing, transition rates

$$n-1 \xleftarrow{\frac{n}{\varepsilon}} n \xrightarrow{\frac{n}{\varepsilon}} n+1$$

for $n = 1,2,\ldots$, and initial state $D(0,\varepsilon,\tau_\ell)$ which has Poisson$(\ell/(2\varepsilon))$ distribution.

In more detail, the number $D(v, v+\varepsilon, \tau_\ell)$ is the number of branches at level v in a critical binary $(0,\varepsilon)$ branching process started with a Poisson$(\ell/(2\varepsilon))$

number of initial individuals. From the Poisson distribution of $D(0,\varepsilon,\tau_\ell)$, and the law of large numbers,

$$\lim_{\varepsilon\downarrow 0}\varepsilon D(0,\varepsilon,\tau_\ell)=\ell \qquad \text{almost surely}$$

and similarly, for each $v>0$ and $\ell>0$, by consideration of excursions of R away from level v

$$\lim_{\varepsilon\downarrow 0}2\varepsilon D(v,v+\varepsilon,\tau_\ell)=L^v_{\tau_\ell}(R) \qquad \text{almost surely.}$$

This process $(2\varepsilon D(v,v+\varepsilon,\tau_\ell),v\geq 0)$, which serves as an approximation to $(L^v_{\tau_\ell}(R),v\geq 0)$, is a Markov chain whose state space is the set of integer multiples of 2ε, with transition rates

$$x-2\varepsilon \;\xleftarrow{\frac{x}{2\varepsilon^2}}\; x \;\xrightarrow{\frac{x}{2\varepsilon^2}}\; x+2\varepsilon$$

for $x=2\varepsilon n>0$. The generator G_ε of this Markov chain acts on smooth functions f on $(0,\infty)$ according to

$$\begin{aligned}(G_\varepsilon f)(x)&=\frac{x}{2\varepsilon^2}f(x-2\varepsilon)+\frac{x}{2\varepsilon^2}f(x+2\varepsilon)-\frac{x}{\varepsilon^2}f(x)\\&=4x\,\frac{1}{(2\varepsilon)^2}\left[\tfrac12 f(x-2\varepsilon)+\tfrac12 f(x+2\varepsilon)-f(x)\right]\\&\to 4x\,\frac12\frac{d^2}{dx^2}f \text{ as } \varepsilon\to 0.\end{aligned}$$

Hence, appealing to a suitable approximation of diffusions by Markov chains [239, 238], we obtain the following *Ray-Knight theorem* (Ray [365], Knight [220]):

For each fixed $\ell>0$, and $\tau_\ell:=\inf\{t:L^0_t(R)>\ell\}$, where $R=|B|$,

$$(L^v_{\tau_\ell}(R),v\geq 0)\overset{d}{=}(X^{(0)}_{\ell,v},v\geq 0) \tag{81}$$

where $(X^{(\delta)}_{\ell,v},v\geq 0)$ for $\delta\geq 0$ denotes a squared Bessel process of dimension δ started at $\ell\geq 0$, as in Section 4.4.2. Moreover, if $T_\ell:=\tau_{2\ell}:=\inf\{t>0: L^0_t(B)=\ell\}$, the the processes $(L^v_{T_\ell}(B),v\geq 0)$ and $(L^{-v}_{T_\ell}(B),v\geq 0)$ are two independent copies of $(X^{(0)}_{\ell,v},v\geq 0)$. The squared Bessel processes and their bridges, especially for $\delta=0,2,4$, are involved in the description of the local time processes of numerous Brownian path fragments [220, 365, 437, 350]. For instance, if $T_1:=\inf\{t:B_t=1\}$, then according to Ray and Knight

$$(L^v_{T_1}(B),0\leq v\leq 1)\overset{d}{=}(X^{(2)}_{0,1-v},0\leq v\leq 1). \tag{82}$$

Many proofs, variations and extensions of these basic Ray-Knight theorems can be found in the literature. See for instance [212, 370, 348, 426, 190] and papers cited there. The appearance of squared Bessel processes processes embedded in the local times of Brownian motion is best understood in terms of the construction of these processes as weak limits of Galton-Watson branching processes with immigration, and their consequent interpretation as continuous state branching processes with immigration [212]. For instance, there is the following expression of the Lévy-Itô representation of squared Bessel processes, and its interpretation in terms of Brownian excursions [350], due to Le Gall-Yor [257]: For R a reflecting Brownian motion on $[0, \infty)$, with $R_0 = 0$, let

$$Y_t^{(\delta)} := R_t + L_t^0(R)/\delta \quad (t \geq 0).$$

Then for $\delta > 0$ the process of ultimate local times of $Y^{(\delta)}$ is a squared Bessel process of dimension δ started at 0:

$$(L_\infty^v(Y^{(\delta)}), v \geq 0) \stackrel{d}{=} (X_{0,v}^{(\delta)}, v \geq 0). \tag{83}$$

See [349, §8] for further discussion of these results and their explanation in terms of random trees embedded in Brownian excursions.

6.3. Additive functionals

A process $(F_t, t \geq 0)$ derived from the path of a process X is called an *additive functional* if for all $s, t \geq 0$

$$F_{s+t}(X_u, u \geq 0) = F_s(X_u, u \geq 0) + F_t(X_{s+u}, u \geq 0)$$

almost surely. Basic additive functionals of any process X are the integrals

$$F_t = \int_0^t f(X_s) ds \tag{84}$$

for suitable f. For each $x \in \mathbb{R}$, the local time process $(L_t^x, t \geq 0)$ is an additive functional of a one-dimensional Brownian motion B. McKean and Tanaka [297] showed that for $X = B$ a one-dimensional Brownian motion, every continuous additive functional of locally bounded variation can be represented as

$$F_t = \int \mu(dx) L_t^x$$

for some signed Radon measure μ on $\mathbb{R}$. According to the occupation density formula (74), the case $\mu(dx) = f(x)dx$ reduces to (84) for $X = B$. For a d-dimensional Brownian motion with $d \geq 2$ there is no such representation in terms of local times. However each additive functional of bounded variation can be associated with a signed measures on the state space, called its *Revuz measure* [368] [369] [130] [419]. Another kind of additive functional is obtained from the stochastic integral

$$G_t = \int_0^t g(B_s) \cdot dB_s$$

where the integrand is a function of B_s. Such martingale and local martingale additive functionals were studied by Ventcel' [424] for Brownian motion and by Motoo, Kunita and Watanabe [237] [314] for more general Markov processes.

6.4. Quadratic functionals

By a *quadratic Brownian functional*, we mean primarily a functional of the form

$$\int \mu(ds) B_s^2$$

for some positive measure $\mu(ds)$ on $\mathbb{R}_+$. But it is also of interest to consider the more general functionals

$$\int \mu(ds) \left(\int_0^\infty f(s,t) dB_t \right)^2$$

for μ and f such that

$$\int \mu(ds) \int_0^\infty dt\, f^2(s,t) < \infty.$$

In terms of the Wiener chaos decomposition (63), these functionals belong to $C_0 \bigoplus C_2$. So in full generality, we use the term *quadratic Brownian functional* to mean any functional of the form

$$c + \int_0^\infty dB_s \int_0^s dB_u \phi(s,u)$$

with $c \in \mathbb{R}$ and $\int_0^\infty ds \int_0^s du \phi^2(s,u) < \infty$. We note that, with the help of Kahunen-Loéve expansions, the laws of such functionals may be decribed via their characteristic functions. These may be expanded as infinite products, which can sometimes be evaluated explicitly in terms of hyperbolic functions or other special functions. See e.g. Neveu [322] Hitsuda [170]. Perhaps the most famous example is Lévy's stochastic area formula

$$\begin{aligned} \mathbb{E}\left[\exp\left(i\lambda \int_0^t (X_s dY_s - Y_s dX_s)\right)\right] &= \mathbb{E}\left[\exp - \left(\frac{\lambda^2}{2} \int_0^t ds (X_s^2 + Y_s^2)\right)\right] \\ &= \frac{1}{\cosh(\lambda t)} \end{aligned} \tag{85}$$

where X and Y are two independent standard BMs. See Lévy[270] Gaveau [149] Berthuet [22] Biane-Yor [38] for many variations of this formula, some of which are reviewed in Yor [453].

A number of noteworthy identities in law between quadratic Brownian functionals are consequences of the following elementary observation:

$$\int_0^\infty ds \left(\int_0^\infty f(s,t) dB_t \right)^2 \stackrel{d}{=} \int_0^\infty ds \left(\int_0^\infty f(t,s) dB_t \right)^2$$

for $f \in L^2(\mathbb{R}_+^2; ds\, dt)$. Consequences of this observation include the following identity, which was discovered by chemists studying the radius of gyration of random polymers

$$\int_0^1 ds \left(B_s - \int_0^1 du B_u \right)^2 \stackrel{d}{=} \int_0^1 ds\, (B_s - sB_1)^2$$

where the left side involves centering at mean value of the Brownian path on $[0, 1]$, while the right side involves a Brownian bridge. The right side is known in empirical process theory to describe the asymptotic distribution of the von Mises statistic [391]. More generally Yor [452] explains how the Cieselski-Taylor identities, which relate the laws of occupation times and hitting times of Brownian motion in various dimensions, may be understood in terms of such identities in law between two quadratic Brownian functionals. See also [451] and [290, Ch. 4].

6.5. Exponential functionals

Some references on this topic are [456] [292] [293] [163] [457].

7. Path decompositions and excursion theory

A basic technique in the analysis of Brownian functionals, especially additive functionals, is to decompose the Brownian path into various fragments, and to express the functional of interest in terms of these path fragments. Application of this technique demands an adequate description of the joint distribution of the *pre-ρ* and *post-ρ* fragments

$$(B_t, 0 \le t \le \rho) \text{ and } (B_{\rho+s}, 0 \le s < \infty)$$

for various random times ρ. If ρ is a stopping time, then according to the strong Markov property these two fragments are conditionally independent given B_ρ, and the post ρ process is a Brownian motion with random initial state B_ρ. But the strong Markov property says nothing about the distribution of the pre-ρ fragment. More generally, it is of interest to consider decompositions of the Brownian path into three or more fragments defined by cutting at two or more random times.

7.1. Brownian bridge, meander and excursion

To facilitate description of the random path fragment of random length, the following notation is very convenient. For a process $X := (X_t, t \in J)$ parameterized by an interval J, and $I = [G_I, D_I]$ a random subinterval of J with length $\lambda_I := D_I - G_I > 0$, we denote by $X[I]$ or $X[G_I, D_I]$ the *fragment of X on I*, that is the process

$$X[I]_u := X_{G_I+u} \qquad (0 \le u \le \lambda_I). \tag{86}$$

We denote by $X_*[I]$ or $X_*[G_I, D_I]$ the *standardized fragment of X on I*, defined by the *Brownian scaling operation*

$$X_*[I]_u := \frac{X_{G_I+u\lambda_I} - X_{G_I}}{\sqrt{\lambda_I}} \qquad (\, 0 \le u \le 1). \tag{87}$$

Note that the fundamental invariance of Brownian motion under Brownian scaling can be stated in this notation as

$$B_*[0,T] \stackrel{d}{=} B[0,1]$$

for each fixed time $T > 0$. Let $G_T := \sup\{s : s \leq T, B_s = 0\}$ be the last zero of B before time T and $D_T := \inf\{s : s > T, B_s = 0\}$ be the first zero of B after time T. Let $|B| := (|B_t|, t \geq 0)$, called *reflecting Brownian motion*. It is well known [180, 76, 370] that there are the following identities in distribution derived by Brownian scaling: for each fixed $T > 0$

$$B_*[0,G_T] \stackrel{d}{=} B^{\mathrm{br}} \tag{88}$$

where B^{br} is a *standard Brownian bridge*,

$$|B|_*[G_T,T] \stackrel{d}{=} B^{\mathrm{me}} \tag{89}$$

where B^{me} is a *standard Brownian meander*, and

$$|B|_*[G_T,D_T] \stackrel{d}{=} B^{\mathrm{ex}}. \tag{90}$$

where B^{ex} is a *standard Brownian excursion*. These identities in distribution provide a convenient unified definition of the standard bridge, meander and excursion, which arise also as limits in distribution of conditioned random walks, as discussed in Section 2. It is also known that $B^{\mathrm{br}}, B^{\mathrm{me}}$ and B^{ex} can be constructed by various other operations on the paths of B, and transformed from one to another by further operations [25].

The excursion straddling a fixed time For each fixed $T > 0$, the path of B on $[0,T]$ can be reconstructed in an obvious way from the four random elements

$$G_T,\ B_*[0,G_T],\ |B|_*[G_T,1],\ \mathrm{sign}(B_T)$$

which are independent, the first with distribution

$$\mathbb{P}(G_T/T \in du) = \frac{du}{\pi\sqrt{u(1-u)}} \qquad (0 < u < 1)$$

which is one of Lévy's arc-sine laws, the next a standard bridge, the next a standard meander, and the last a uniform random sign $\pm\frac{1}{2}$. Similarly, the path of B on $[0,D_T]$ can be reconstructed from the four random elements

$$(G_T,D_T),\ B_*[0,G_T],\ |B|_*[G_T,D_T],\ \mathrm{sign}(B_T)$$

which are independent, with the joint law of (G_T,D_T) given by

$$\mathbb{P}(G_T/T \in du, D_T/T \in dv) = \frac{du\,dv}{2\pi u^{1/2}(v-u)^{3/2}} \qquad (0 < u < 1 < v),$$

with $B_*[0, G_T]$ a standard bridge, $|B|_*[G_T, D_T]$ a standard excursion, and $\operatorname{sign}(B_T)$ a uniform random sign $\pm\frac{1}{2}$. See [448, Chapter 7].

For $0 < t < \infty$ let $B^{\mathrm{br},t}$ be a *Brownian bridge of length* t, which may be regarded as a random element of $C[0, t]$ or of $C[0, \infty]$, as convenient:

$$B^{\mathrm{br},t}(s) := \sqrt{t} B^{\mathrm{br}}((s/t) \wedge 1) \qquad (s \geq 0). \tag{91}$$

Let $B^{\mathrm{me},t}$ denote a *Brownian meander of length* t, and $B^{\mathrm{ex},t}$ be a *Brownian excursion of length* t, defined similarly to (91) with B^{me} or B^{ex} instead of B^{br}.

Brownian excursions and the three-dimensional Bessel process There is a close connection between Brownian excursions and a particular time-homogeneous diffusion process R_3 on $[0, \infty)$, commonly known as the *three-dimensional Bessel process* BES(3), due to the representation

$$(R_3(t), t \geq 0) \stackrel{d}{=} \left(\sqrt{\sum_{i=1}^{3} (B_i(t))^2}, t \geq 0 \right) \tag{92}$$

where the B_i are three independent standard Brownian motions. It should be understood however that this particular representation of R_3 is a relatively unimportant coincidence in distribution. What is more important, and can be understood entirely in terms of the random walk approximations of Brownian motion and Brownian excursion (1) and (7), is that there exists a time-homogeneous diffusion process R_3 on $[0, \infty)$ with $R_3(0) = 0$, which has the same self-similarity property as B, meaning invariance under Brownian scaling, and which can be characterized in various ways, including (92), but most importantly as a Doob h-transform of Brownian motion.

For each fixed $t > 0$, the Brownian excursion $B^{\mathrm{ex},t}$ of length t is the BES(3) bridge from 0 to 0 over time t, meaning that

$$(B^{\mathrm{ex},t}(s), 0 \leq s \leq t) \stackrel{d}{=} (R_3(s), 0 \leq s \leq t \,|\, R_3(t) = 0).$$

Moreover, as $t \to \infty$

$$B^{\mathrm{ex},t} \stackrel{\mathrm{d}}{\to} R_3, \tag{93}$$

and R_3 can be characterized in two other ways as follows:

(i) [294, 437] The process R_3 is a Brownian motion on $[0, \infty)$ started at 0 and conditioned never to return to 0, as defined by the Doob h-transform, for the harmonic function $h(x) = x$ of Brownian motion on $[0, \infty)$, with absorbtion at 0. That is, R_3 has continuous paths starting at 0, and for each $0 < a < b$ the stretch of R_3 between when it first hits a and first hits b is distributed like B with $B_0 = a$ conditioned to hit b before 0.

(ii) [347] There is the identity

$$R_3(t) = B(t) - 2\underline{B}(t) \qquad (t \geq 0) \tag{94}$$

where B is a standard Brownian motion with past minimum process

$$\underline{B}(t) := \underline{B}[0, t] = -\underline{R_3}[t, \infty).$$

The identity in distribution (94) admits numerous variations and conditioned forms [347, 25, 31]. For instance, by application of Lévy's identity (78)

$$(R_3(t), t \geq 0) \stackrel{d}{=} (|B_t| + L_t, t \geq 0). \tag{95}$$

where $(L_t, t \geq 0)$ is the local time process of B at 0.

7.2. The Brownian zero set

Consider the *zero set* of one-dimensional Brownian motion B:

$$Z(\omega) := \{t : B_t(\omega) = 0\}.$$

Since B has continuous paths, $Z(\omega)$ is closed subset of $[0, \infty)$, which depends on ω through the path of B. Intuitively, $Z(\omega)$ is a random closed subset of $[0, \infty)$, and this is made rigorous by putting an appropriate σ-field on the set of all closed subsets of $[0, \infty)$. Some almost sure properties of the Brownian zero are:

- $Z(\omega)$ has Lebesgue measure equal to 0;
- $Z(\omega)$ has Hausdorff dimension $1/2$;
- $Z(\omega)$ has no isolated points;
- $Z(\omega)$ is the set of points of increase of the local time process at 0.
- $Z(\omega)$ is the closure of the range of the inverse local time process, which is a stable subordinator of index $1/2$.

See [355] for a study of the distribution of ranked lengths of component open intervals of $(0, t) \setminus Z(\omega)$, and generalizations to a stable subordinator of index $\alpha \in (0, 1)$.

7.3. Lévy-Itô theory of Brownian excursions

The Lévy-Itô excursion theory allows the Brownian path to be reconstructed from its random zero set, an ensemble of independent standard Brownian excursions, and a collection of independent random signs, one for each excursion. The zero set can first be created as the closed range of a stable subordinator $(T_\ell, \ell \geq 0)$ which ends up being the inverse local time process of B. Then for each ℓ such that $T_{\ell-} < T_\ell$ the path of B on $[T_{\ell-}, T_\ell]$ can be recreated by shifting and scaling a standard Brownian excursion to start at time $T_{\ell-}$ and end at time T_ℓ.

Due to (78), the process of excursions of $|B|$ away from 0 is equivalent in distribution to the process of excursions of B above $\underline{B}$. According to the Lévy-Itô description of this process, if $I_\ell := [T_{\ell-}, T_\ell]$ for $T_\ell := \inf\{t : B(t) < -\ell\}$, the points

$$\{(\ell, \mu(I_\ell), (B - \underline{B})[I_\ell]) : \ell > 0, \mu(I_\ell) > 0\}, \tag{96}$$

where μ is Lebesgue measure, are the points of a Poisson point process on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C[0, \infty)$ with intensity

$$d\ell \, \frac{dt}{\sqrt{2\pi}\, t^{3/2}} \, \mathbb{P}(B^{\mathrm{ex},t} \in d\omega). \tag{97}$$

On the other hand, according to Williams [438], if $M_\ell := \overline{B}[I_\ell] - \underline{B}[I_\ell]$ is the maximum height of the excursion of B over $\underline{B}$ on the interval I_ℓ, the points

$$\{(\ell, M_\ell, (B - \underline{B})[I_\ell]) : \ell > 0, \mu(I_\ell) > 0\}, \tag{98}$$

are the points of a Poisson point process on $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C[0,\infty)$ with intensity

$$d\ell \, \frac{dm}{m^2} \, \mathbb{P}(B^{\mathrm{ex}\,|\,m} \in d\omega) \tag{99}$$

where $B^{\mathrm{ex}\,|\,m}$ is a *Brownian excursion conditioned to have maximum* m. That is to say $B^{\mathrm{ex}\,|\,m}$ is a process X with $X(0) = 0$ such that for each $m > 0$, and $H_x(X) := \inf\{t : t > 0, X(t) = x\}$, the processes $X[0, H_m(X)]$ and $m - X[H_m(X), H_0(X)]$ are two independent copies of $R_3[0, H_m(R_3)]$, and X is stopped at 0 at time $H_0(X)$. *Itô's law of Brownian excursions* is the σ-finite measure ν on $C[0,\infty)$ which can be presented in two different ways according to (97) and (99) as

$$\nu(\cdot) = \int_0^\infty \frac{dt}{\sqrt{2\pi} t^{3/2}} \, \mathbb{P}(B^{\mathrm{ex},t} \in \cdot) = \int_0^\infty \frac{dm}{m^2} \, \mathbb{P}(B^{\mathrm{ex}\,|\,m} \in \cdot) \tag{100}$$

where the first expression is a disintegration according to the lifetime of the excursion, and the second according to its maximum. The identity (100) has a number of interesting applications and generalizations [36, 351, 356]. See [370, Ch. XII], [352] and [448] for more detailed accounts of Itô's excursion theory and its applications.

Notes and Comments See [371, 252, 23, 370, 160] for different approaches to the basic path transformation (94) from B to R_3, its discrete analogs, and various extensions. In terms of $X := -B$ and $M := \overline{X} = -\underline{B}$, the transformation takes X to $2M - X$. For a generalization to exponential functionals, see Matsumoto and Yor [291]. This is also discussed in [333], where an alternative proof is given using reversibility and symmetry arguments, with an application to a certain directed polymer problem. A multidimensional extension is presented in [334], where a representation for Brownian motion conditioned never to exit a (type A) Weyl chamber is obtained using reversibility and symmetry properties of certain queueing networks. See also [333, 228] and the survey paper [332]. This representation theorem is closely connected to random matrices, Young tableaux, the Robinson-Schensted-Knuth correspondence, and symmetric functions theory [331, 335]. A similar representation theorem has been obtained in [58] in a more general symmetric spaces context, using quite different methods. These multidimensional versions of the transformation from X to $2M - X$ are intimately connected with combinatorial representation theory and Littelmann's path model [278].

8. Planar Brownian motion

8.1. Conformal invariance

Lévy showed that if

$$Z_t := B_t^{(1)} + iB_t^{(2)}$$

is a 2-dimensional Brownian motion, regarded here as a $\mathbb{C}$-valued process, and $f : \mathbb{C} \to \mathbb{C}$ is a non-constant holomorphic function, then

$$f(Z_t) = \widehat{Z}\left(\int_0^t |f'(Z_s)|^2 ds\right) \tag{101}$$

where $(\widehat{Z}(u), u \geq 0)$ is another $\mathbb{C}$-valued Brownian motion. This is an instance of the Dambis-Dubins-Schwarz result of Section 5.6.3 in two dimensions. For

$$f(Z_t) = F_t^{(1)} + iF_t^{(2)}$$

for real-valued processes $F^{(1)}$ and $F^{(2)}$ which are two local martingales relative to the filtration of (Z_t), with

$$\langle F^{(1)}\rangle_t = \langle F^{(2)}\rangle_t \text{ and } \langle F^{(1)}, F^{(2)}\rangle_t \equiv 0$$

as a consequence of Itô's formula and the Cauchy-Riemann equations for f.

More generally, Getoor and Sharpe [151] defined a *conformal martingale* to be any $\mathbb{C}$-valued continuous local martingale $(Z_t := X_t + iY_t, t \geq 0)$ for real-valued X and Y such that

$$\langle X\rangle_t = \langle Y\rangle_t \text{ and } \langle X, Y\rangle_t \equiv 0.$$

They showed in this setting that

$$Z_t = \widehat{Z}_{\langle X\rangle_t} \qquad (t \geq 0)$$

where $(\widehat{Z}_u, u \geq 0)$ is a $\mathbb{C}$-valued Brownian motion. Note that conformal martingales are stable by composition with an entire holomorphic function, and also by continuous time-changes. See [104], [91] [313] for many applications.

8.2. Polarity of points, and windings

According to Lévy's theorem,

$$\exp(Z_t) = \widehat{Z}\left(\int_0^t ds \exp(2X_s)\right) \tag{102}$$

with $(\widehat{Z}(u), u \geq 0)$ another planar BM started at 1, and X_s the real part of Z_s. It follows immediately from (102) that $\widehat{Z}$ will never visit 0 almost surely. From this, it follows easily that for two arbitrary points z_0 and z_1 with $z_0 \neq z_1$

$$\mathbb{P}^{z_0}(Z_t = z_1 \text{ for some } t \geq 0) = 0.$$

In particular, the winding number process, that is a continuous determination $(\theta_t^{(z_1)}, t \geq 0)$ of the argument of $Z_t - z_1$ along the path of Z, is almost surely well defined for all $t \geq 0$, and so is the corresponding complex logarithm of $Z_t - z_1$, according to the formula

$$\log(|Z_t - z_1|) + i\theta_t^{(z_1)} := \int_0^t \frac{dZ_u}{Z_u - z_1} = \tilde{Z}\left(\int_0^t \frac{du}{|Z_u - z_1|^2}\right) \tag{103}$$

for $t \geq 0$, where, by another application of Lévy's theorem, the process $\tilde{Z}$ is a complex Brownian motion starting at 0. Moreover, from the trivial identity

$$Z_t - z_1 = (z_0 - z_1) + \int_0^t \frac{dZ_u}{(Z_u - z_1)}(Z_u - z_1)$$

considered as a linear integral equation in $(Z_u - z_1)$, we see that

$$Z_t - z_1 = (z_0 - z_1) \cdot \exp\left(\int_0^t \frac{dZ_u}{(Z_u - z_1)}\right).$$

Taking $z_1 = 0$, this yields the *skew-product representation* of the planar BM Z started from $z_0 \neq 0$:

$$Z_t = |Z_t| \exp\left[i\gamma\left(\int_0^t \frac{ds}{|Z_s|^2}\right)\right] \tag{104}$$

for $t \geq 0$, where $(\gamma(u), u \geq 0)$ is a one dimensional BM, independent of the radial part $(|Z_t|, t \geq 0)$, which is by definition a 2-dimensional Bessel process. This skew-product representation reduces a number of problems involving a planar Brownian motion Z to problems involving just its radial part.

8.3. Asymptotic laws

In 1958, the following two basic asymptotic laws of planar Brownian motion were discovered independently:

Spitzer's law [395]

$$\frac{2\theta_t}{\log t} \xrightarrow{\mathrm{d}} C_1 \text{ as } t \to \infty$$

where θ_t denotes the winding number of Z around 0, assuming $Z_0 \neq 0$, and C_1 denotes a standard Cauchy variable.

The Kallianpur-Robbins law [203]

$$\frac{1}{(\log t)||f||} \int_0^t ds f(Z_s) \xrightarrow{\mathrm{d}} e_1 \text{ as } t \to \infty$$

for all non-negative measurable functions f with $||f|| := \int\int f(x+iy)dxdy < \infty$, and e_1 a standard exponential variable, along with the ratio ergodic theorem

$$\frac{\int_0^t dsf(Z_s)}{\int_0^t dsg(Z_s)} \overset{\text{a.s.}}{\to} \frac{||f||}{||g||} \text{ as } t \to \infty$$

for two such functions f and g. It was shown in [353] that the skew product representation of planar BM allows a unified derivation of these asymptotic laws, along with descriptions of the joint asymptotic behaviour of windings about several points and an additive functions

$$\frac{1}{\log t}\left(\theta_t^{(z_1)}, \ldots, \theta_t^{(z_k)}, \int_0^t dsf(Z_s)\right).$$

For a number of extensions of these results, and a review of literature around this theme, see Yor [453, Ch. 8]. See also Watanabe [431] Franchi [138] Le Gall and Yor [262] [258] [256] for various extensions of these asymptotic laws: to more general recurrent diffusion processes in the plane, to Brownian motion on Riemann surfaces, and to windings of Brownian motion about lines and curves in higher dimensions. See also [449, Chapter 7] where some asymptotics are obtained for the self-linking number of BM in $\mathbb{R}^3$. Questions about knotting and entanglement of Brownian paths and random walks are studied in [214] and [318]. Much more about planar Brownian motion can be found in Le Gall's course [259].

8.4. Self-intersections

In a series of remarkable papers in the 1950's, Dvoretsky, Erdős, Kakutani and Taylor established among other things the existence of multiple points of arbitary (even infinite) order in planar Brownian paths. It was not until the 1970's that any attempt was made to quantify the extent of self-intersection of Brownian paths by consideration of the occupation measure of Brownian increments

$$\nu_{s,t}^{(2)}(\omega, dx) := \int_0^s du \int_0^t dv\, 1(B_u - B_v \in dx). \tag{105}$$

for a planar Brownian motion B, and $x \in \mathbb{R}^2$. Wolpert [447] and Rosen [374] showed that this random measure is almost surely absolutely continuous with respect to Lebsesgue measure on $\mathbb{R}^2$, with a density

$$\tilde{\alpha}(x; s, t), x \in \mathbb{R}^2 - \{0\}, s, t, \geq 0), \tag{106}$$

which can be chosen to be jointly continuous in (x, s, t). For fixed x this process in s, t is called the the process of *intersection local times at* x. However,

$$\lim_{x\to 0} \tilde{\alpha}(x; s, t) = \infty \text{ almost surely}$$

reflecting the accumulation of immediate intersections of the Brownian path with itself coming from times u, v in (105) with u close to v. A measure of the extent of such self-intersections is obtained by consideration of

$$\nu_{s,t}^{(2)} f_n := \int_{\mathbb{R}^2} f_n(x) \nu_{s,t}^{(2)}(\omega, dx) = \int_0^s du \int_0^t dv f_n(B_u - B_v)$$

as $n \to \infty$ with $f_n(x) := n^2 f(nx)$ for a continuous non-negative function f with compact support and planar Lebesgue integral equal to 1. Varadhan [422] showed that

$$\nu_{s,t}^{(2)} f_n - \mathbb{E}(\nu_{s,t}^{(2)} f_n) \to \gamma_{s,t} \text{ as } n \to \infty$$

in every L^p space for some limit $\gamma_{s,t}$ which is independent of f. Indeed,

$$\tilde{\alpha}(x; s, t) - \mathbb{E}[\tilde{\alpha}(x; s, t)] \to \gamma_{s,t} \text{ as } x \to 0$$

in L^2. The limit process $(\gamma_{s,t}, s, t \geq 0)$ is known to admit a continuous version in (s, t), called the *renormalized self-intersection local time* of planar Brownian motion. Rosen [375] obtained variants of Tanaka's formula for these local times of intersection.

These results have been extended in a number of ways. In particular, for each $k > 2$ the occupation measure of Brownian increments of order $k - 1$

$$\int_0^{s_1} du_1 \cdots \int_0^{s_k} du_k \prod_{i=2}^{k} (B_{u_i} - B_{u_{i-1}} \in dx_i)$$

is absolutely continuous with respect to the Lebesgue measure $dx_2 \cdots dx_k$, and Varadhan's renormalization result extends as follows: the multiple integrals

$$\int \cdots \int_{0 \leq s_1 < \cdots < s_k \leq t} ds_1 \cdots ds_k \prod_{i=2}^{k} \overline{f_n(B_{u_i} - B_{u_{i-1}})},$$

where $\overline{X} := X - \mathbb{E}(X)$, converge in L^p for every $p \geq 1$ as $n \to \infty$, to define a kth order renormalized intersection local time. Amongst a number of deep applications of these intersection local times, we mention the asymptotic series expansion of the area of the *Wiener sausage*

$$S_\epsilon(t) := \{y : |y - B_s| \leq \varepsilon \text{ for some } 0 \leq s \leq t\},$$

according to which for each fixed $n = 1, 2, \ldots$, as $\epsilon \to 0$

$$m(S_\epsilon(t)) = \sum_{k=1}^{n} \frac{\gamma_k(t)}{(\log(1/\epsilon))^k} + o\left(\frac{1}{(\log(1/\epsilon))^n}\right)$$

where $\gamma_k(t)$ is the kth order normalized intersection local time. These results, and much more in the same vein, can be found in the course of Le Gall [263]. Subsequent open problems were collected in [102].

Questions about self-intersection of planar Brownian paths are closely related to questions about intersections of two or more independent Brownian paths. Such questions are easier, because a process $L^{(1,2)}_{t,s}$ of local time of intersection at 0 between two independent Brownian paths $(B^{(1)}_u, 0 \le u \le t)$ and $(B^{(2)}_v, 0 \le v \le s)$ is well-defined and finite. Le Gall's proof of Varadhan's renormalization uses the convergence of a centered sequence of such local times of intersection $L^{(1,2)}$.

8.5. Exponents of non-intersection

Another circle of questions involves estimates of the probability of non-intersection of independent planar BM's $B^{(1)}$ and $B^{(2)}$. If $Z^{(i)}_R$ is the range of the path of $B^{(i)}_t$ for $0 \le t \le \inf\{t : |B_t| = R\}$, and $B^{(i)}_0$ is uniformly distributed on the unit circle, then there exist two universal constants c_1 and c_2 such that

$$c_1 R^{-5/4} \le \mathbb{P}(Z^{(1)}_R \cap Z^{(2)}_R = \emptyset) \le c_2 R^{-5/4}. \tag{107}$$

More generally, for p independent BM's, there is a corresponding exponent

$$\xi_p = (4p^2 - 1)/12$$

instead of 5/4, so that

$$c_{1,p} R^{-\xi_p} \le \mathbb{P}\left(\cap_{i=1}^p Z^{(i)}_R = \emptyset\right) \le c_{2,p} R^{-\xi_p}$$

Another interesting family of exponents, the *non-disconnection exponents* η_p are defined by

$$c'_{1,p} R^{-\xi_p} \le \mathbb{P}\left(\mathcal{D}(\cup_{i=1}^p Z^{(i)}_R)\right) \le c'_{2,p} R^{-\xi_p}$$

where $\mathcal{D}(\Gamma)$ is the event that the random set Γ does not disconnect 0 from ∞. It is known that these exponents exist, and are given by the formula

$$\eta_p = \frac{(\sqrt{24p+1} - 1)^2 - 4}{48}.$$

In particular, $\eta_1 = 1/4$. For an account of these results, see Werner [434, p. 165, Theorem 8.5]. These results are the keys to computation of the Hausdorff dimension $d_H(\Gamma)$ of a number of interesting random sets Γ derived from a planar Brownian path, in particular

- the set C of *cut points* B_t for t such that $B[0,t] \cap B[t,1] = \emptyset$:

$$d_H(C) = 2 - \xi_1 = 3/4$$

- the set F of *frontier points* B_t for t such that $\mathcal{D}(B[0,1] - B_t)$:

$$d_H(F) = 2 - \eta_2 = 4/3$$

- the set P of *pioneer points* B_t such that $\mathcal{D}(B[0,t]-B_t)$:

$$d_H(P) = 2 - \eta_1 = 7/4.$$

In particular, the second of these evaluations established a famous conjecture of Mandelbrot. For introductions to these ideas and the closely related theory of *Schramm-Loewner evolutions* driven by Brownian motion, see Le Gall's review [261] of the work of Kenyon, Lawler and Werner on critical exponents for random walks and Brownian motion, the appendix of [313] by Schramm and Werner, and the monograph of Lawler [250] on conformally invariant processes in the plane,

9. Multidimensional BM

9.1. Skew product representation

The skew product representation (104) of planar Brownian motion can be generalized as follows to a BM B in $\mathbb{R}^d$ for $d \geq 2$:

$$B_t = |B_t|\theta(\textstyle\int_0^t ds/|B_s|^2), \qquad t \geq 0 \tag{108}$$

where the radial process $(|B_t|, t \geq 0)$ is a d-dimensional Bessel process, as discussed in Section 4.4.2, and the angular Brownian motion $(\theta_u = \theta(u), u \geq 0)$ is a BM on the sphere $\mathbb{S}^{d-1}$, independent of the radial process. This is a particular case of BM on a manifold, as discussed further in Section 10. For now, we just indicate Stroock's representation of the angular motion $(\theta_u, u \geq 0)$ as the solution of the Stratonovich differential equation

$$\theta_u^i = \theta_0^i + \sum_{j=1}^d \int_0^u (\delta_{ij} - \theta_s^i\theta_s^j) \circ dW_s^j, \qquad (1 \leq i \leq d) \tag{109}$$

where (W_s) is a d-dimensional BM independent of $(|B_s|)$. This representation (109) may be obtained by application of Itô's formula to $f(x) = x/|x|$ and Knight's theorem of Section 5.6.4 on orthogonal martingales.

9.2. Transience

The BM in $\mathbb{R}^d$ is transient for $d \geq 3$, and the study of its rate of escape to ∞ relies largely on the application of one-dimensional diffusion theory to the radial process. See for instance Shiga and Watanabe [387]. According to the theory of last exit decompositions, the *escape process*

$$S_r := \sup\{t, |B_t| = r\}, \qquad r \geq 0$$

is a process with independent increments, whose distribution can be described quite explicitly [150]. In the special case $d = 3$ this process is the stable process of index $\frac{1}{2}$, with stationary independent increments, and is identical in law to $(T_r, r \geq 0)$ where T_r is the first hitting time of r by a one-dimensional Brownian motion B. This and other coincidences involving one-dimensional Brownian motion and the three-dimensional Bessel process BES(3) are explained by the fact that BES(3) is the Doob h-transform of BM on $(0, \infty)$ with killing at 0.

9.3. Other geometric aspects

Studies of stochastic integrals such as

$$\int_0^t (B_s^i dB_s^j - B_s^j dB_s^i), \quad i \neq j$$

arise naturally in various contexts, such as Brownian motion on the Heisenberg group. See e.g. Gaveau [149] and Berthuet [22].

9.3.1. Self-intersections

Dvoretsky, Erdős and Kakutani [107] showed that two independent BM's in $\mathbb{R}^3$ intersect almost surely, even if started apart. Consequently, a single Brownian path in $\mathbb{R}^3$ has self-intersections almost surely. An occupation measure of Brownian increments $\nu_{s,t}^{(3)}$ can be defined exactly as in (105), for B with values in $\mathbb{R}^3$ instead of $\mathbb{R}^2$, and this measure admits a density $\tilde{\alpha}(x; s, t), x \in \mathbb{R}^3 - \{0\}, s, t, \geq 0$ which may be chosen jointly continuous in (x, s, t). Again $\lim_{x \to 0} \tilde{\alpha}(x; s, t) = \infty$ almost surely, but now Varadhan's renormalization phenomenon does not occur. Rather, there is a weaker result. In agreement with Symanzik's program for quantum field theory, Westwater [435] showed weak convergence as $n \to \infty$ of the measures

$$\frac{\exp\left(-g\nu_{s,t}^{(3)}(f_n)\right)}{Z_n^g(s,t)} \cdot W^{(3)} \tag{110}$$

where $g > 0$, $W^{(3)}$ is the Wiener measure,

$$\nu_{s,t}^{(3)}(f_n) = \int_{\mathbb{R}^3} f_n(x) \nu_{s,t}^{(3)}(\omega; dx) = \int_0^s du \int_0^t dv f_n(B_u - B_v)$$

where $f_n(x) := n^3 f(nx)$ for a continuous non-negative function f with compact support and Lebesgue integral equal to 1. Westwater showed that as g varies the weak limits $W^{(3,g)_{s,t}}$ are mutually singular, and that under each $W^{(3,g)_{s,t}}$ the new process, while no longer a BM, still has self-intersections.

For $\delta \geq 4$, two independent BM's in $\mathbb{R}^\delta$ do not intersect, and consequently BM($\mathbb{R}^\delta$) has no self-intersections. The analog of (110) can nonetheless be studied, with the result that these measures still have weak limits. Some references are [54] [60] [64] [178].

9.4. Multidimensional diffusions

Basic references on this subject are:

[411] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes* (1979).
[408] D. W. Stroock. *Lectures on stochastic analysis: diffusion theory* (1987).
[423] S. R. S. Varadhan. *Diffusion processes* (2001).
[432] S. Watanabe. *Itô's stochastic calculus and its applications* (2001).

See also [210, Chapter 5], [336][15]. Basic notions are weak and strong solutions of stochastic differential equations, and martingale problems. We refer to Platen [358] for a survey of literature on numerical methods for SDE's, including Euler approximations, stochastic Taylor expansions, multiple Itô and Stratonovich integrals, strong and weak approximation methods, Monte Carlo simulations and variance reduction techniques for functionals of diffusion processes. An earlier text in this area is Kloeden and Platen [219].

9.5. Matrix-valued diffusions

Another interesting example of BM in higher dimensional spaces is provided by matrix-valued BM's, which are of increasing interest in the theory of random matrices. See for instance O'Connell's survey [332]. Cépa and Lépingle [70] interpret Dyson's model for the eigenvalues of $N \times N$ unitary random random matrices as a system of N Brownian interacting Brownian particles on the circle with electrostatic repulsion. They discuss more general particle systems allowing collisions between particles, and measure-valued limits of such systems as $N \to \infty$. This relates to the asymptotic theory of random matrices, which concerns asymptotic features as $N \to \infty$ of various statistics of $N \times N$ matrix ensembles. See also [63]. Another interesting development from random matrix theory is the theory of Dyson's Brownian motions [398] [428].

9.6. Boundaries

9.6.1. Absorbtion

It is natural in many contexts to consider Brownian motion and diffusions in some connected open subset D of $\mathbb{R}^N$. The simplest such process is obtained *absorbing* B when it first reaches the boundary at time $T_D = \inf\{t \geq 0 : B_t \notin D\}$. This gives a Markov process X, with state space the closure of D, defined by $X_t = B_{t \wedge T_D}$. A key connection with classical analysis is provided by considering the density of the mean occupation measure of the killed BM: for all non-negative Borel measurable functions f vanishing off D

$$\mathbb{E}^x \int_0^\infty f(X_t)dt = \mathbb{E}^x \int_0^{T_D} f(B_t)dt = \int_D f(y) g_D(x,y)dy \tag{111}$$

where $g_D(x, y)$ is the classical *Green function* associated with the domain D. A classical fact, not particularly obvious from (111), is that g_D is a symmetric function of (x, y). See Mörters and Peres [313, §3.3] and Chung [77] for further discussion.

9.6.2. Reflection

If H is the half-space of $\mathbb{R}^N$ on one side of a hyperplane, there is an obvious way to create a process R with continuous paths in the closure of H by reflection through the hyperplane of a Brownian motion in $\mathbb{R}^N$. By Skorokhod's analysis of reflecting Brownian motion on $[0, \infty)$ in the one-dimensional case, this reflecting Brownian motion R in H can be described as a semi-martingale, which is the sum of a Brownian motion and process of bounded variation which adds a push only when R is on the boundary hyperplane, in a direction normal to the hyperplane, and of precisely the right magnitude to keep R on one side of the hyperplane. This semi-martingale description has been generalized to characterize reflecting BM in a convex polytope bounded by any finite number of hyperplanes, and further to domains with smooth boundaries. See for instance [277] [446] [209] [393] [65, Ch. 5] A basic property of such reflecting Brownian motions R is that in great generality Lebesgue measure on the domain is the unique invariant measure. In particular, if D is compact, as $t \to \infty$ the limiting distribution of R_t is uniform on D. More complex boundary behaviour is possible, and of interest in applications of reflecting BM to queuing theory [446] and the study of Schramm-Loewner evolutions [250, Appendix C].

9.6.3. Other boundary conditions

See Ikeda-Watanabe [175] for a general discussion of boundary conditions for diffusions, with references to the large Japanese literature of papers by Sato, Ueno, Motoo and others dating back to the 60's.

10. Brownian motion on manifolds

10.1. Constructions

A Riemannian manifold M is a manifold equipped with a Riemannian metric. Starting from this structure, there are various expressions for the Laplace-Beltrami operator Δ, and the Levi-Civita connection. Closely associated with the Laplace-Beltrami operator is the fundamental solution of the heat equation on M derived from $\frac{1}{2}\Delta$. This defines a semigroup of transition probability operators from which one can construct a Brownian motion on M. Alternatively, the Brownian motion on M with generator $\frac{1}{2}\Delta$ can be constructed by solving a martingale problem associated with $\frac{1}{2}\Delta$. Note that in general the possibility of explosion must be allowed: the M-valued Brownian motion B may be defined only up to some random *explosion time* $e(B)$.

At least two other constructions of Brownian motion on M may be considered, one known as *extrinsic*, the other as *intrinsic*. Some examples of the extrinsic construction appear in the work of Lewis and van den Berg [272] [420]. In general, this construction relies on Nash's embedding of M as a submanifold of $\mathbb{R}^\ell$, with the induced metric. Following Hsu [171, Ch. 3], let $\{\xi_\alpha, 1 \leq \alpha \leq \ell\}$ be the standard orthonormal basis in $\mathbb{R}^\ell$, let P_α be the orthogonal projection of ξ_α onto T_xM, the tangent space at $x \in M$. Then P_α is a vector field on M, and the Laplace-Beltrami operator Δ can be written as

$$\Delta = \sum_{\alpha=1}^{\ell} P_\alpha^2$$

and the Brownian motion started at $x \in M$ may be constructed as the solution of the Stratonovich SDE

$$dX_t = \sum_\alpha P_\alpha(X_t) \circ dW_t^\alpha, \qquad X_0 = x \in M.$$

where $(W^\alpha, 1 \leq \alpha \leq \ell)$ is a BM in $\mathbb{R}^\ell$. See also Rogers and Williams [373] and Stroock [409, Ch. 4] for further development of the extrinsic approach.

The intrinsic approach to construction of BM on a manifold involves a lot more differential geometry. See Stroock [409, Chapters 7 and 8] and other texts listed in the references, which include the theory of semimartingales on manifolds, as developed by L. Schwartz, P. A. Meyer and M. Emery.

10.2. Radial processes

Pick a point $o \in M$, a Riemannian manifold of dimension 2 or more, and with the help of the exponential map based at o, define polar coordinates (r, θ) and hence processes $r(B_t)$ and $\theta(B_t)$ where B is Brownian motion on M. The *cutlocus of* o, denoted C_o, is a subset of M such that r is smooth on $M - \{o\} - C_o$, the set $M - C_o$ is a dense open subset of M, and the distance from o to C_o is positive. Kendall [215] showed that there exists a one-dimensional Brownian motion β and a non-decreasing process L, the local time process at the cutlocus, which increases only when $B_t \in C_o$ such that

$$r(X_t) - r(X_0) = \beta_t + \tfrac{1}{2} \int_0^t \Delta\, r(B_s) ds - L_t \qquad t < e(B).$$

See also [216], [171, Th. 3.5.1] and [339]. Hsu [171, §4.2], shows how this representation of the radial process allows a comparison with a one-dimensional diffusion process to conclude that a growth condition on the lower bound of the Ricci curvature provides a sufficient condition for the BM not to explode. The condition is also necessary under a further regularity condition on M.

10.3. References

We refer to the following monographs and survey articles for further study of Brownian motion and diffusions on manifolds.

Monographs

[115] K. D. Elworthy. *Stochastic differential equations on manifolds* (1982).
[114] David Elworthy. *Geometric aspects of diffusions on manifolds* (1988).
[116] K. D. Elworthy, Y. Le Jan, and Xue-Mei Li. *On the geometry of diffusion operators and stochastic flows* (1999).
[118] M. Émery. *Stochastic calculus in manifolds* (1989).
[171] Elton P. Hsu. *Stochastic analysis on manifolds* (2002).
[175] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes* (1989).
[283] Paul Malliavin. *Géométrie différentielle stochastique* (1978).
[303] P.-A. Meyer *Géometrie différentielle stochastique. II* (1982).
[373] L. C. G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales, Vol. II: Itô Calculus* (1987).
[380] L. Schwartz. *Géométrie différentielle du 2ème ordre, semi-martingales et équations différentielles stochastiques sur une variété différentielle.* (1982).
[381] L. Schwartz. *Semimartingales and their stochastic calculus on manifolds* (1984).
[409] D. W. Stroock. *An introduction to the analysis of paths on a Riemannian manifold* (2000).

Survey articles

[311] S. A. Molčanov. *Diffusion processes, and Riemannian geometry* (1975).
[302] P.-A. Meyer. *A differential geometric formalism for the Itô calculus* (1981)

11. Infinite dimensional diffusions

[207] G. Kallianpur and J. Xiong *Stochastic differential equations in infinite-dimensional spaces* (1995).

11.1. Filtering theory

We refer to the textbook treatments of [336, Ch. VI] and [373, Ch. VI] Rogers-Williams and the monographs

[204] G. Kallianpur *Stochastic filtering theory* (1980).
[202] G. Kallianpur and R. L. Karandikar *White noise theory of prediction, filtering and smoothing* (1988)

See also the survey papers by Kunita [234] [235].

11.2. Measure-valued diffusions and Brownian superprocesses

Some monographs are

[93] D.A. Dawson. *Measure-valued Markov processes* (1994).

[111] E. B. Dynkin. *Diffusions, superdiffusions and partial differential equations* (2002).
[121] A. M. Etheridge *An introduction to superprocesses* (2000).
[253] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations* (1999).
[315] B. Mselati. *Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation* (2004).

These studies are related to the non-linear PDE $\Delta u = u^2$. See also Dynkin [112] who treats the equation $\Delta u = u^\alpha$ for $1 < \alpha \leq 2$ and Le Gall [254].

11.3. Malliavin calculus

The term *Malliavin calculus*, formerly called *stochastic calculus of variations*, refers to the study of infinite-dimensional, Gaussian probability spaces inspired by P. Malliavin's paper [284]. The calculus is designed to study the probability densities of functionals of Gaussian processes, hypoellipticity of partial differential operators, and the theory of non-adapted stochastic integrals. The following account is quoted from Ocone's review of Nualart's text [326] in *Math. Reviews*:

> A partial differential operator A is hypoelliptic if u is C^∞ on those open sets where Au is C^∞. Hörmander's famous hypoellipticity theorem states that if $A = X_0 + \sum_1^d X_i^2$, where $X_0, \cdots, X_d$ are smooth vector fields, and if the Lie algebra generated by $X_0, \cdots, X_d$ is full rank at all points, then A is hypoelliptic. Now, second-order operators such as A appear as infinitesimal generators of diffusion processes solving stochastic differential equations driven by Brownian motion. Thus the study of A can be linked to the theory of stochastic differential equations (SDEs). In particular, hypoellipticity of the generator is connected to the existence of smooth densities for the probability laws of the solution. If the Fokker-Planck operator is hypoelliptic, the solutions of the Fokker-Planck equation are smooth functions providing transition probability densities for the corresponding diffusion. Conversely, it is possible to work back from the existence of smooth densities to hypoellipticity. Because solutions of SDEs are functionals of the driving Brownian motion, the question of hypoellipticity of the generator is then an aspect of a much more general problem. Given an $\mathbb{R}^n$-valued functional $G(W)$ of a Gaussian process W, when does the probability distribution of $G(W)$ admit a density with respect to Lebesgue measure and how regular is it? Malliavin realized how to approach this question using a differential calculus for Wiener functionals. His original work contained two major achievements: a general criterion for the existence and regularity of probability densities for functionals of a Gaussian process, and its application to solutions of SDEs, leading to a fully stochastic proof of Hörmander's theorem.
> Malliavin's paper was tremendously influential, because it provided stochastic analysts with a genuinely new tool. Of the major lines of investigation that ensued, let us mention the following, in only the roughest manner, as background to Nualart's new text. First is the continued study of existence and representation of densities of Wiener functionals, its application to hypoellipticity, short-time asymptotics, index theorems, etc. of solutions to second-order operator equations, and its application to solutions of infinite-dimensional stochastic evolution equations, such as stochastic PDE, interacting particle systems, or delay equations. The seminal work here is due to Stroock, Kusuoka, Watanabe, and Bismut. Second, Wiener space calculus has found application to the quite different problem of analysis of non-adapted Brownian functionals, following a paper of Nualart

and Pardoux, who derived a calculus for non-adapted stochastic integrals, using heavily the Sobolev spaces defined in Wiener space analysis. This made possible a new study of stochastic evolution systems in which anticipation of the driving noise occurs or in which there is no flow of information given by a filtration. Finally, we mention that the Malliavin calculus has led to new progress and applications of the problem of quasi-invariance of Wiener processes under translation by nonlinear operators, a non-adapted version of the Girsanov problem.

General references

[17] D. R. Bell. *The Malliavin calculus* (1987).
[49] J.-M. Bismut. *Large deviations and the Malliavin calculus* (1984).
[175] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes* (1989).
[285] P. Malliavin. *Stochastic analysis* (1997).
[324] J. Norris. *Simplified Malliavin calculus* (1986).
[326] D. Nualart. *The Malliavin calculus and related topics* (1995).
[405] and
[406] . D. W. Stroock. *The Malliavin calculus and its application to second order parabolic differential equations, I and II* (1981). See also [404] and [407]]. [429] S. Watanabe *Lectures on stochastic differential equations and Malliavin calculus* (1984).
[441] D. Williams. *To begin at the beginning: ...* (1981).

Applications to mathematical finance

[88] *Conference on Applications of Malliavin Calculus in Finance* (2003)
[286] P. Malliavin and A. Thalmaier. *Stochastic calculus of variations in mathematical finance* (2006).

12. Connections with analysis

Kahane [197] provides a historical review of a century of interplay between Taylor series, Fourier series and Brownian motion.

12.1. Partial differential equations

For general background on PDE, we refer to L. C. Evans [123]. The texts of Katazas and Shreve [210, Chapter 4], Freidlin [140] and Durrett [Ch. 8][104] all contain material on connections between BM and PDE. Other sources are Bass [14] and Doob [99], especially for parabolic equations, and Glover [153].

12.1.1. Laplace's equation: harmonic functions

We begin by considering Laplace's equation:

$$\Delta u = 0.$$

where $\Delta u := \sum_{i=1}^n u_{x_i,x_i}$. A *harmonic function* is a C^2 function u which solves Laplace's equation.

Let $D(x,r) := \{y : |y-x| < r\}$ and let D be a *domain*, that is a connected open subset of $\mathbb{R}^n$. Let $\tau_D := \inf\{t : B_t \in D^c\}$. Since each component of a Brownian motion B in $\mathbb{R}^n$ is a.s. unbounded, $P(\tau_D < \infty) = 1$ for any bounded domain D. If u is harmonic in D, then

$$u(x) = \int_{\partial D(x,r)} u(y) S(dy) \tag{112}$$

for every $x \in D$ and $r > 0$ such that $\overline{D(x,r)} \subset D$, where S is the uniform probability distribution on $\partial D(x,r)$ which is invariant under orthogonal transformations. To see this, observe that by Itô's formula,

$$\begin{aligned} u(B_{t\wedge\tau_{D(x,r)}}) &= u(x) + \sum_{i=1}^n \int_0^{t\wedge\tau_{D(x,r)}} u_{x_i}(B_s) dB_s + \frac{1}{2}\int_0^{t\wedge\tau_{D(x,r)}} \Delta u(B_s) ds \\ &= u(x) + \sum_{i=1}^n \int_0^{t\wedge\tau_{D(x,r)}} u_{x_i}(B_s) dB_s \end{aligned}$$

which is a continuous local martingale. But since $u(B_{t\wedge\tau_{D(x,r)}}) - u(x)$ is uniformly bounded, it is a true martingale with mean 0. Taking expectation and using $P(\tau_{D(x,r)} < \infty) = 1$ gives

$$u(x) = \mathbb{E}^x\left(u(B_{\tau_{D(x,r)}})\right) = \int_{\partial D(x,r)} u(y) dS.$$

where the last equality is by the symmetry of Brownian motion with respect to orthogonal transformations. Conversely, if $u : D \to \mathbb{R}$ has this mean value property, then u is C^∞ and harmonic. (Gauss's theorem).

12.1.2. The Dirichlet problem

Some references are [210, §4.2], [14], [104], [313]. Consider the equation

$$\Delta u = 0, \text{ in } D \text{ and } u = f \text{ on } \partial D \tag{113}$$

where $\mathbb{E}^x(|f(B_{\tau_D})|) < \infty$. Then $u(x) := \mathbb{E}_x(f(B_{\tau_D}))$ has $\Delta u = 0$ in D. because

$$\begin{aligned} u(x) = \mathbb{E}_x f(B_{\tau_D}) &= \mathbb{E}_x\left(\mathbb{E}_x\left(f(B_{\tau_D})|\mathcal{F}_{\tau_{D(x,r)}}\right)\right) \\ &= \mathbb{E}_x u(B_{\tau_{D(x,r)}}) \quad \text{by strong Markov property} \\ &= \int_{\partial D(x,r)} u(y) dS \end{aligned}$$

So in order to have a solution to the partial differential equation (113), we need:

$$\lim_{x\to a} \mathbb{E}_x\left(f(B_{\tau_D})\right) = f(a), \quad a \in \partial D.$$

This is true under a natural condition on the boundary. It should be *regular*:

$$\text{if } \sigma_D = \inf\{t > 0 : B_t \in D^c\}, \text{ then } \mathbb{P}_x(\sigma_D = 0) = 1, \quad \forall x \in \partial D.$$

See [175] for a more refined discussion, treating irregular boundaries.

12.1.3. Parabolic equations

Following is a list of PDE's of parabolic type related to BM($\mathbb{R}^\delta$). For $u : [0,\infty) \times \mathbb{R}^\delta \to \mathbb{R}$, write $u = u(t,x)$, let $u_t := \partial u/\partial t$, and let Δ denote the Laplacian, and ∇ the gradient, acting on the variable x. Assume the initial boundary condition $u(0,x) = f(x)$. Then

$$u_t = \tfrac{1}{2}\Delta u \tag{114}$$

is solved by

$$u(t,x) = \mathbb{E}^x[f(B_t)].$$

$$u_t = \tfrac{1}{2}\Delta u + g \tag{115}$$

for $g : [0,\infty) \times \mathbb{R}^\delta \to \mathbb{R}$ is solved by

$$u(t,x) = \mathbb{E}^x\left[f(B_t) + \int_0^t g(t-s, B_s)ds\right].$$

$$u_t = \tfrac{1}{2}\Delta u + cu \tag{116}$$

where $c = c(x) \in \mathbb{R}$ is solved by

$$u(t,x) = \mathbb{E}^x\left[f(B_t)\exp\left(\int_0^t c(B_s)ds\right)\right]. \tag{117}$$

$$u_t = \tfrac{1}{2}\Delta u + b \cdot \Delta u \tag{118}$$

for $b = b(x) \in \mathbb{R}$ is solved by

$$u(t,x) = \mathbb{E}^x\left[f(B_t)\exp\left(\int_0^t b(B_s) \cdot dB_s - \tfrac{1}{2}\int_0^t |b(B_s)|^2 ds\right)\right]. \tag{119}$$

Equation (114) is the classical *heat equation*, discussed already in Section 4.3. See [210, §4.3] for the one-dimensional case, and Doob [99] for a more extensive discussion.

Equation (118) is the variant when Brownian motion B is replaced by a BM with drift b, which may be realized as the solution of the SDE

$$X_t = x + B_t + \int_0^t b(X_s)ds. \tag{120}$$

It is a nice result, due to Zvonkin [459], for dimension $\delta = 1$ and Veretennikov-Krylov [425] for $\delta = 2, 3, \ldots$, that for b Borel bounded, equation (120) has a unique strong solution; that the solution is unique in law is a consequence of Girsanov's theorem. The right side of (119) equals $\mathbb{E}^x[f(X_t)]$ for (X_t) the solution of (120).

The expression on the right side of (117) is the celebrated *Feynman-Kac formula*. See [217] for a historical review, and [189] for further discussion of the case $\delta = 1$ using Brownian excursion theory to analyse

$$\int_0^\infty dt e^{-\lambda t} \mathbb{E}^x \left[f(B_t) \exp\left(\int_0^t c(B_s) ds \right) \right].$$

A well known application of the Feynman-Kac formula is Kac's derivation of Lévy's arcsine law for the distribution of $\int_0^t 1(B_s > 0) ds$ for B a BM($\mathbb{R}$). Kac also studied the law of $\int_0^t |B_s| ds$ by the same method. See Biane-Yor [37] and [342] for related results. Extensions of the Feynman-Kac formula to more general Markov processes are discussed in [372] [357] [131].

12.1.4. The Neumann problem

Just as the distribution of B stopped when it first hits ∂D solves the Dirichlet boundary value problem, for suitably regular domains D the transition function $p(t, x, y)$ of B reflected in ∂D is the fundamental solution of the Neumann boundary value problem:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2} \Delta_x \right) p(t, x, y) = 0 \qquad (t > 0, x, y \in D) \tag{121}$$

with the Neumann boundary condition

$$\frac{\partial}{\partial n_x} p(t, x, y) = 0 \qquad (t > 0, x \in D, y \in D) \tag{122}$$

where n_x is the inner normal at the point $x \in \partial D$, and initial condition

$$\lim_{t \downarrow 0} p(t, x, y) = \delta_y(x) \qquad (x, y \in D). \tag{123}$$

See Fukushima [143], Davies [90], Ikeda-Watanabe [175], Hsu [172] and Burdzy [65]. See [16] for treatment of related problems for irregular domains.

12.1.5. Non-linear problems

Le Gall's monograph [260] treats spatial branching processes and random snakes derived from Brownian motion, and their relation to non-linear partial differential equations such as $\Delta u = u^2$.

12.2. Stochastic differential Equations

In order to consider more general PDEs, we need to introduce the notion of a *stochastic differential equations (SDEs)*. We say that the semimartingale X solves the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

if

$$X_t = X_0 + \int_o^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \tag{124}$$

Solutions to three equations exist in particular when σ and b are bounded and Lipschitz. The proof is based on Picard's iteration.

Claim 1. *If X_t solves (124), then*

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X)ds, \ t \geq 0, \ f \in C^2$$

where

$$Af(x) = 1/2 \sum_{i,j=1}^n a^{ij}(x) f_{x_i x_j}(x) + \sum_{i=1}^n b^i(x) f_{x_i}(x)$$

and $a = \sigma\sigma^T$, is a martingale.

Proof.

$$\begin{aligned} < X^i, X^j > &= \sum_{k,l=1}^n < \sigma^{ik}(X).B^k, \sigma^{jl}(X).B^l >_t \\ &= \sum_{k,l=1}^n \sigma^{ik}\sigma^{jl}. < B^k, B^l >_t \\ &= \int_0^t a^{ij} < X_s > ds. \end{aligned}$$

So by Itô's formula,

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^n \int_0^t f_{X_i}(X)dX^i + 1/2 \sum_{i,j=1}^n \int_0^t f_{x_i x_j}(X_s)d < X^i, X^j >_s \\ &= f(X_0) + \sum_{i,j=1}^n \int_0^t \sigma^{ij}(X_s) f_{x_i}(X_s)dB_s^j + \int_0^t Af(X)ds \end{aligned}$$

Now assume that $u \in C^2(D) \cap C(\overline{D})$ is a solution of

$$-Au(X) = f(X) \text{ in } D, \text{ and } u = 0 \text{ on } \partial D.$$

Then $u(x) = \mathbb{E}_x \left(\int_0^{\tau_D} f(X_s) ds \right)$. By Itô,

$$\begin{aligned} u(X_{t \wedge \tau_D}) - u(x) &= M^f_{t \wedge \tau_D} + \int_0^{t \wedge \tau_D} Au(X_s) ds \\ &= M^f_{t \wedge \tau_D} - \int_0^{t \wedge \tau_D} f(X_s) ds \end{aligned}$$

Now taking expectation and limit as $t \to \infty$,

$$\mathbb{E}_x u(X_{\tau_D}) - u(x) = -\mathbb{E}_x \int_0^{\tau_D} f(X_s) ds$$

and so

$$u(x) = \mathbb{E}_x \int_0^{\tau_D} f(X_s) ds.$$

□

12.2.1. Dynamic Equations

We consider *dynamic equations* of the form

$$\begin{aligned} u_t &= Au - cu \text{ in } (0, \infty) \times \mathbb{R}^n \\ u(0, x) &= f(X) \end{aligned}$$

We show that the C^2 solutions of this equation are of the form

$$u(t, x) = \mathbb{E}_x \left(f(X_t) exp \int_0^t c(X_s) ds \right)$$

The first step is to show that $u(t-s, X_s) exp \left(- \int_0^t c(X_s) ds \right)$ is a local martingale on $[0, t)$.

If c, u is bounded, then M_s above is a bounded martingale. The martingale convergence theorem implies that as $s \nearrow t$, $M_s \to M_t$. Since u is continuous and $u(0, x) = f(x)$, we must have

$$lim_{s \nearrow t} M_s = f(B_t) exp \left(- \int_0^t c(X_s) ds \right).$$

So we have

$$\mathbb{E}_x f(X_t) exp \left(- \int_0^t c(X_s) ds \right) = u(t, x)$$

We have seen that the solution to

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

is

$$u(x) = \mathbb{E}_x \int_0^{\tau_D} f(B_s) ds.$$

So if $f = 1$, we have $\mathbb{E}_x(\tau_D)$ is the solution of

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

For example, if $D = B(0,1)$, then the solution is $(1 - |x|^2)/n$, $\Rightarrow$ $\mathbb{E}_x(\tau_{D(x,1)}) = (1 - |x|^2)/n$.

12.3. Potential theory

The probabilistic theory of Brownian motion is closely related to the classical potential theory of the Laplace operator and the parabolic potential theory of the heat operator.

Analytic treatments of potential theory

[213] O. D. Kellogg. *Foundations of potential theory* (1929).
[167] L. L. Helms. *Introduction to potential theory* (1969).
[6] D. H. Armitage and S. J. Gardiner *Classical potential theory* (2001).
[433] N. A. Watson. *Introduction to heat potential theory* (2012).

Relations between Brownian motion and potential theory

[196] M. Kac. *Aspects probabilistes de la théorie du potentiel* (1970).
[195] M. Kac. *Integration in function spaces and some of its applications* (1980).
[359] S. C. Port and C. J. Stone. *Brownian motion and classical potential theory* (1978).
[364] M. Rao. *Brownian motion and classical potential theory* (1977).
[396] F. Spitzer. *Electrostatic capacity, heat flow, and Brownian motion* (1964).
[394] R. Durrett and H. Kesten, editors. *Random walks, Brownian motion, and interacting particle systems* (1991).
[99] J. L. Doob. *Classical potential theory and its probabilistic counterpart* (1984).

12.4. BM and harmonic functions

Some references are:

[14] R. F. Bass. *Probabilistic techniques in analysis* (1995).
[15] R. F. Bass. *Diffusions and elliptic operators* (1998).
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[92] B. Davis. *Brownian motion and analytic functions* (1979).
[104] R. Durrett. *Brownian motion and martingales in analysis* (1984).
[159] R. F. Gundy. *Some topics in probability and analysis* (1989).
[198] J.-P. Kahane. *Brownian motion and classical analysis* (1976).
[344] K. E. Petersen. *Brownian motion, Hardy spaces and bounded mean oscillation* (1977).

12.5. Hypercontractive inequalities

Hypercontractive estimates show that some semigroup (P_t) acting on a probability space is a contraction form L^p to L^q with $p < q$, and some $t > 0$. Neveu [320] used stochastic integration with respect to Brownian motion to establish a hypercontractive estimate due to Nelson for conditional expectations of Gaussian variables. Bakry and Émery [9] treat hypercontractive diffusions and their relation to the logarithmic Sobolev inequalities of Gross [158]. See also Stroock [403], Bañuelos and Davis [7]. Saloff-Coste [377] relates Poincaré, Sobolev, and Harnack inequalities in the setting of a second order partial differential operator on a manifold. Ledoux [264] [265] offers overviews of the broader domain of isoperimetry and Gaussian analysis.

13. Connections with number theory

According to the central limit theorem of Erdös–Kac [120] in the theory of additive number theoretic functions, if $\omega(n)$ is the number of distinct prime factors of n, then for n picked uniformly at random from the integers from 1 to N, as $N \to \infty$ the limit distribution of $(\omega(n) - \log\log n)/\sqrt{\log\log n}$ is standard normal. Billingsley [45] showed how Brownian motion appears as the limit distribution of a random path created by a natural extension of this construction. See also Philipp [346] and Tennenbaum [417].

The articles of Williams [443] and Biane, Pitman and Yor [40] study various Brownian functionals whose probability densities involve Jacobi's theta functions, and whose Mellin transforms involve the Riemann zeta function and other zeta functions that arise in analytic number theory. In particular, the well known functional equation satisfied by the Riemann zeta function involves moments of the common distribution of the range of a Brownian bridge and the maximum of a Brownian excursion. See also [363] regarding other probability distributions related to the Riemann zeta function, and further references.

14. Connections with enumerative combinatorics

There are several contexts in which Brownian motion, or some conditioned fragment of Brownian motion like Brownian bridge or Brownian excursion, arises in a natural way as the limit distribution of a random path created from a random combinatorial object of size n as $n \to \infty$. In the first instance, as in Section 2, these limit processes are obtained from a path with uniform distribution on 2^n paths of length n with increments of $\pm$, or on a suitable subset of such paths. Less obviously, Brownian excursion also arises as the limit distribution of a path encoding the structure of any one of a number of natural combinatorial models of random trees. Harris [164] may have been the first to recognize the branching structure encoded in a random walk. This structure was exploited by Knight [220] in his analysis of the local time process of Brownian motion defined by limits of occupation times of random walks. Aldous [3] developed the concept of the *Brownian continuum random tree*, now understood as a particular kind of random real tree [124] that is naturally encoded in the path of a Brownian excursion. Aldous showed how Brownian excursion arises as the weak limit of a height profile process derived derived from a Galton-Watson tree conditioned to have n vertices, as $n \to \infty$, for any offspring distribution with finite variance. This includes a number of natural combinatorial models for trees with n vertices. See [349] and [124] for reviews of the work of Aldous and others on this topic.

For a uniformly distributed random mapping from an n element set to itself, Aldous and Pitman [4] showed how encoding trees in the digraph of the mapping as excursions of a random walk leads to a functional limit theorem in which a reflecting Brownian bridge is obtained as the limit. This yields a large number of limit theorems for attributes of the random mapping such as height of the tallest tree in the mapping digraph. See [349, §9] and [124] for an overview and further references, and [349, §6.4] for a brief account of Aldous's theory of critical random graphs and the multiplicative coalescent.

Much richer limiting continuum structures related to the Brownian continuum tree have been derived in the last decade in the analysis of limit distributions for random planar maps and related processes. Some recent articles on this topic are

[308] G. Miermont *Random maps and continuum random 2-dimensional geometries* (2013).
[255] J.-F. Le Gall *The Brownian map: a universal limit for random planar maps* (2014).
[87] N. Curien and J.-F. Le Gall *Scaling limits for the peeling process on random maps* (2017).
[30] J. Bertoin, and N. Curien and I. Kortchemski. *Random planar maps and growth-fragmentations* (2018).

14.1. Brownian motion on fractals

Some articles are

[11] M. T. Barlow and R. F. Bass. *Brownian motion and harmonic analysis on Sierpinski carpets* (1999).
[12] M. T. Barlow, R. F. Bass, T. Kumagai and A. Teplyaev. *Uniqueness of Brownian motion on Sierpiński carpets* (2010).

[231] T. Kumagai *Estimates of transition densities for Brownian motion on nested fractals* (1993)
[232] T. Kumagai. *Brownian motion penetrating fractals: an application of the trace theorem of Besov spaces* (2000).

14.2. Analysis of fractals

Lalley [243] relates Brownian motion in the plane to the equilibrium measure on the Julia set of a rational mapping. For analytic treatment of related results about harmonic measure for domains with complicated boundaries, including Makarov's theorem on harmonic measure [282], see Garnett and Marchall [147].

14.3. Free probability

Some references are:

[39] Ph. Biane *Free Brownian motion, free stochastic calculus and random matrices* (1997).
[309] J. A. Mingo and R. Speicher *Free probability and random matrices* (2017).

15. Applications

15.1. Economics and finance

Some historical sources are Bachelier's thesis [8], Black and Scholes [50]. Some textbooks:

[137] J.-P. Fouque, G. Papanicolaou and K. R. Sircar. *Derivatives in financial markets with stochastic volatility*, (2000).
[244] D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance* (2008).
[286] P. Malliavin and A. Thalmaier. *Stochastic calculus of variations in mathematical finance* (2006).
[399] J. M. Steele. *Stochastic calculus and financial applications* (2001).
[205] G. Kallianpur and R. K. Karandikar. *Introduction to option pricing theory* (2000).

See also Karatzas-Shreve [210, §5.8].

The theory of optimal stopping in continuous time is developed by Øksendal [336, Ch. X]. Two monographs on this topic are:

[388] A. N. Shiryaev *Optimal stopping rules* (1978).
[343] G. Peskir and A. Shiryaev *Optimal stopping and free-boundary problems* (2006).

15.2. Statistics

The application of Brownian motion to the theory of empirical processes is treated in [391] and [33]. Siegmund [392] treats Brownian motion approximations of the sequential probability ratio test. This kind of application has motivated many studies of boundary crossing probabilities for Brownian motion, such as [269].

Some texts on statistical inference for diffusion processes and fractional Brownian motions are [32] [142] [361] [242] [360].

15.3. Physics

Some basic references are Einstein's 1905 paper [113], and Nelson's book [319]. See also the translation [267] of Paul Langevins 1908 paper [248] on Brownian motion. Redner [366] provides a physical perspective on first-passage processes. Hammond [162] offers a recent review of Smoluchowski's theory of coagulation and diffusion. Sme general texts on applications of stochastic processes in physical sciences are

[421] N. G. van Kampen *Stochastic processes in physics and chemistry* (1981).
[146] C. Gardiner. *Stochastic methods: A handbook for the natural and social sciences* (2009).

Brownian motion has played an important role in the development of various models of physical processes involving random environments and random media. The Brownian map mentioned in Section 14 may be regarded as providing a random two-dimensional geometry. Closely related studies are the theory of the Gaussian free field and Liouville quantum gravity. Some monographs in this vein are

[412] A.-S. Sznitman. *Brownian motion, obstacles and random media* (1998).
[55] E. Bolthausen and A.-S. Sznitman. *Ten lectures on random media* (2002).
[136] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna. *Wave propagation and time reversal in randomly layered media* (2007).
[413] A.-S. Sznitman. *Topics in occupation times and Gaussian free fields* (2012).
[458] L. Zambotti. *Random obstacle problems* (2017).

Other themes related to diffusions in random environments are treated in:

[119] J. Engländer *Spatial branching in random environments and with interaction* (2015).
[367] P. Révész *Random walk in random and nonrandom environments* (1990).
[227] T. Komorowski, C. Landim and S. Olla *Fluctuations in Markov processes* (2012).
[386] Z. Shi *Sinai's walk via stochastic calculus*, (2001).

15.4. Fluid mechanics

Systems of interacting Brownian motions have been used as a computational tool in fluid mechanics. The Navier-Stokes equations and the Prandtl boundary layer equations of fluid mechanics can be interepreted as Fokker-Planck (or Kolmogorov) equations for interacting particles which diffuse by Brownian motion and carry vorticity (=rotation). At high Reynolds numbers it is computationally more efficient to model and sample the Brownian motions than to solve the original equations. To satisfy the boundary conditions particles have to be created at solid walls by a branching construction. This idea was introduced in A. Chorin in [73] and [74] for the Prandtl equations. A good theoretical treatment can be found in the paper of Goodman [156]See also the book of Cottet and Koumoutsakos [83]for a fairly comprehensive account of the theory and practice of these methods.

15.5. Control of diffusions

An introduction is provided by Øksendal [336, Ch. XI]. We thank Vivek Borkar for providing the following list of monographs on this topic:

[152] Gihman and Skorohod, *Controlled stochastic processes* (1979)
[134] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control* (1975).
[135] W. H. Fleming and H. M. Soner, *Controlled Markov processes and viscosity solutions* (1993)).
[56] V. Borkar, *Optimal control of diffusion processes* (1989)).
[19] A. Bensoussan, *Stochastic control by functional analysis methods* (1982).
[20] A. Bensoussan, *Stochastic control of partially observable systems* (1992).
[230] A. Krylov, *Controlled diffusion processes* (1980)

More recent activity is in applications to finance. See for instance [133] [69] [345] [337]. See [240] regarding numerical methods for stochastic control problems in continuous time.

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[2] Robert J. Adler. *An introduction to continuity, extrema, and related topics for general Gaussian processes.* Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA, 1990.
[3] David Aldous. The continuum random tree. III. *Ann. Probab.*, 21(1):248–289, 1993.
[4] David J. Aldous and Jim Pitman. Brownian bridge asymptotics for random mappings. *Random Structures Algorithms*, 5(4):487–512, 1994.

[5] David Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[6] David H. Armitage and Stephen J. Gardiner. *Classical potential theory*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2001.

[7] Rodrigo Bañuelos and Burgess Davis. A geometrical characterization of intrinsic ultracontractivity for planar domains with boundaries given by the graphs of functions. *Indiana Univ. Math. J.*, 41(4):885–913, 1992.

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[11] Martin T. Barlow and Richard F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canad. J. Math.*, 51(4):673–744, 1999.

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[13] Ole E. Barndorff-Nielsen, Thomas Mikosch, and Sid Resnick, editors. *Lévy processes*. Birkhäuser Boston Inc., Boston, MA, 2001. Theory and applications.

[14] R. F. Bass. *Probabilistic techniques in analysis*. Springer-Verlag, New York, 1995.

[15] R. F. Bass. *Diffusions and elliptic operators*. Springer-Verlag, New York, 1998.

[16] Richard F. Bass, Krzysztof Burdzy, and Zhen-Qing Chen. On the Robin problem in fractal domains. *Proc. Lond. Math. Soc. (3)*, 96(2):273–311, 2008.

[17] Denis R. Bell. *The Malliavin calculus*, volume 34 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1987.

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Theory Related Fields, 73(3):463–480, 1986.

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[33] Rabi Bhattacharya, Lizhen Lin, and Victor Patrangenaru. *A course in mathematical statistics and large sample theory*. Springer Texts in Statistics. Springer-Verlag, New York, 2016.

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