MARKOV CHAINS

What I will talk about in class is pretty close to Durrett Chapter 5 sections 1-5. We stick to the countable state case, except where otherwise mentioned.

Lecture 7. We can regard $(p(i,j))$ as defining a (maybe infinite) matrix $\mathbf{P}$. Then a basic fact is

$$P(X_n = j | X_0 = i) = \mathbf{P}^n(i,j) \tag{12}$$

where $\mathbf{P}^n$ denotes matrix multiplication. There are two conceptually different ways to get this. First regard X_0 as having some arbitrary initial distribution μ_0. Write μ_n for the distribution of X_n:

$$\mu_n(j) = P(X_n = j).$$

By considering the $n+1$'st step we get the forward equation (in vector-matrix notation)

$$\mu_{n+1} = \mu_n \mathbf{P}$$

which implies

$$\mu_n = \mu_0 \mathbf{P}^n.$$

Putting $\mu_0(\cdot) = \delta_i(\cdot)$ gives (12). On the other hand we could fix a bounded function $h : S \to R$ and define

$$h_n(i) = E(h(X_n)|X_0 = i).$$

By considering the first step we get the backward equation

$$h_{n+1} = \mathbf{P} h_n$$

which implies

$$h_n = \mathbf{P}^n h.$$

Putting $h(\cdot) = \delta_j(\cdot)$ gives (12).

Hitting times. Fix $A \subset S$ and consider the first hitting time $\tau_A = \min\{n \geq 0 : X_n \in A\}$. Now consider

$$h_A(i) = P_i(\tau_A < \infty).$$

The next result says h_A is the <u>minimal</u> solution of equations (i)-(iii) below.

(i)$h(i) \geq 0$ for all $i \in S$.

(ii) $h(i) = 1$ for $i \in A$.

(iii) $h(i) = \sum_j p(i,j)h(j)$ for $i \notin A$.

Proposition 18 *(a) h_A satisfies (i)-(iii).*

(b) If h is a solution of (i)-(iii) then $h_A \leq h$.

Proof. (a) is easy. For (b) consider the transition matrix $\mathbf{P}_A$ for "the chain stopped on A"

$$\begin{aligned} p_A(i,j) &= p(i,j), i \notin A \\ &= \delta_i(j), i \in A \end{aligned}$$

Then (ii) and (iii) imply

(iv) $h = \mathbf{P}_A h$

(jargon: h is a <u>harmonic</u> function for $\mathbf{P}_A$.) Fix h satisfying (i)-(iii). Since $h \geq 1_A$,

$$h = \mathbf{P}_A^n h \geq \mathbf{P}_A^n 1_A.$$

This says that $h(i) \geq P_i(X_n^A \in A)$ where X^A denotes a Markov chain with transition matrix $\mathbf{P}_A$. Now given an $(i, \mathbf{P})$-chain (X_n) it is clear that

$$X_n^A = X_{n \wedge \tau_A}$$

defines an $(i, \mathbf{P}^A)$-chain. So

$$h(i) \geq P_i(X_n^A \in A) = P_i(\tau_A \leq n).$$

Let $n \to \infty$.

Lecture 10. In this class we prove the basic results about invariant measures and stationary distributions. The results are those of Durrett Theorems 4.3 - 4.7, though the organization of the proofs is a little different.

Most of the work is in the first result (c.f. Durrett Theorem 4.3: note we don't assume recurrence).

Fix a reference state b and define the <u>b-block occupation measure</u>

$$\mu(b,x) = \sum_{n=0}^{\infty} P_b(X_n = x, T_b > n).$$

Proposition 19 *Consider the equations*

$$\mu(y) = \sum_x \mu(x)p(x,y), y \neq b; \quad \mu(b) = 1,\ 0 \leq \mu(x) \leq \infty \textit{ for all } x. \tag{13}$$

Then $\mu(b,\cdot)$ is the minimal solution of (13). Moreover

$$P_b(T_b < \infty) = \sum_x \mu(b,x)p(x,b).$$

Theorem 20 *Suppose the chain is irreducible and recurrent. Then there exists an invariant measure μ, unique up to scalar multiples. Either*
(i) $\mu(S) = \infty$ and $E_xT_x = \infty$ for all x;
or (ii) $\mu(S) < \infty$ and $E_xT_x < \infty$ for all x.

In case (i) the chain is <u>null-recurrent</u>. In case (ii) the chain is <u>positive-recurrent</u>.

Theorem 21 *Suppose the chain is irreducible. Then it is positive-recurrent iff a stationary distribution π exists. If so, then $\pi(x) = 1/E_xT_x$ for all x.*

Lecture 11. Here is a counterpart to the basic convergence result, Durrett 5.5.

Proposition 22 *Suppose the chain is irreducible but not positive-recurrent. Then $P_\mu(X_n = y) \to 0$ for all y and all initial distributions μ.*

Proof. First, we can reduce to the aperiodic case. Next, the result is obvious in the transient case, because

$$\sum_{n\geq 1} P_\mu(X_n = y) = E_\mu N(y) \leq 1 + E_y N(y) = 1/(1-\rho_{yy}) < \infty.$$

So we may suppose the chain is null-recurrent. Consider independent copies (X_n, Y_n) as a chain on $S \times S$. This product chain is irreducible. If the product chain is transient then as above

$$\sum_{n\geq 1} P_{\mu\times\mu}(X_n = y, Y_n = y) < \infty.$$

But the summands are $(P_\mu(X_n = y))^2$, and these must converge to 0. So suppose the product chain is recurrent. We argue by contradiction. If the result were false, then there exists an initial distribution μ, a state b and a subsequence of times j_n such that

$$P_\mu(X(j_n) = b) \to \alpha_b > 0.$$

By the diagonal argument, we can then pick a subsequence (k_n) such that for every state y

$$P_\mu(X(k_n) = y) \to \alpha_y \geq 0. \tag{14}$$

Now the coupling argument in the proof of Durrett 5.5 depended only on the fact that the product chain was recurrent. So this argument shows that convergence in (14) holds for <u>all</u> initial distributions. The rest of the argument is analysis. Fatou's lemma shows $\sum_y \alpha_y \leq 1$, and then

$$\begin{aligned}
(\alpha\mathbf{P})(y) &= \sum_w (\lim p^{k_n}(x,w))p(w,y) \text{ by (14)} \\
&\leq \lim p^{k_n+1}(x,y) \text{ by Fatou's lemma} \\
&= \lim \sum_z p(x,z)p^{k_n}(z,y) \\
&= \lim \sum_z p(x,z)\alpha_y \text{ by (14) and dominated convergence} \\
&= \alpha_y.
\end{aligned}$$

Now if there really was a strict inequality for some y then

$$\sum_y \alpha_y > \sum_y (\alpha \mathbf{P})(y) = \sum_x \sum_y \alpha_x p(x,y) = \sum_x \alpha_x$$

which is impossible. So we have shown $\alpha = \alpha \mathbf{P}$: this is a finite invariant measure, implying positive-recurrence.

Lecture 12.

The "ergodic theorem" (Durrett 5.1) is a consequence of the SLLN and the following deterministic fact, worth isolating.

Lemma 23 *Let (t_i) be increasing with $n^{-1}t_n \to \bar{t}$. Let*

$$n(t) = \max\{i : t_i \leq t\}.$$

Then $n(t)/t \to 1/\bar{t}$. Now let (r_i) be such that $r_i \geq 0$ and $n^{-1}\sum_{i=1}^n r_i \to \bar{r}$, and let $r(t)$ be such that

$$\sum_{i=1}^{n(t)} r_i \leq r(t) \leq \sum_{i=1}^{n(t)+1} r_i.$$

Then $r(t)/t \to \bar{r}/\bar{t}$.

A number of useful identities may be derived from the following result. For simplicity, suppose our chain is irreducible and finite, aperiodic (really we only need positive-recurrent).

Proposition 24 *Consider the chain started at state x. Let $0 < S < \infty$ be a stopping time such that $X_S = x$. Let y be an arbitrary state. Then*

$$E_x(\textit{number of visits to } y \textit{ before time } S) = \pi(y) E_x S.$$

In the phrase "number of ... before time t", our convention is to include time 0 but exclude time t.

The following series of lemmas arise from particular choices of y and S. Some involve the fundamental matrix

$$Z(x,y) = \sum_{t=0}^{\infty} (p_{xy}^t - \pi(y))$$

(the sum is finite by exercise 5.8).

Lemma 25 *For $y \neq x$,*

$$E_x(\textit{number of visits to } x \textit{ before time } T_y) = \pi(x)(E_xT_y + E_yT_x).$$

Lemma 26

$$E_\pi(\textit{number of visits to } y \textit{ before time } \tau_x)$$

$$= \frac{\pi(y)}{\pi(x)} Z_{xx} - Z_{xy}.$$

Lemma 27 $\pi(x)E_\pi\tau_x = Z_{xx}$.

Lemma 28 $\pi(y)E_x\tau_y = Z_{yy} - Z_{xy}$.