• Recall: We say that a sequence of r.v.’s \((M_n)_{n=0}^\infty\) is a martingale w.r.t. \((Y_n)_{n=0}^\infty\) if \(M_n\) is determined by \((Y_0,\ldots,Y_n)\) for all \(n\) and
\[
\mathbb{E}[M_{n+1} \mid Y_0, Y_1, \ldots, Y_n] = M_n, \text{ for all } n \geq 0.
\] (1)

• Usually we take \(Y_n = M_n\).

• It follows by induction from (1) that martingales have a fixed expectation
\[
\mathbb{E}[M_n] = \mathbb{E}[M_0], \text{ for all } n \geq 0.
\] (2)

• Recall, a \(\mathbb{Z}_+ := \{0, 1, \ldots\}\)-valued random variable \(T\) is called a stopping-time w.r.t. \((Y_n)_{n \geq 0}\) if for all \(n\) the event \(\{T = n\}\) can be determined by \((Y_0, \ldots, Y_n)\) (equivalently, one can use the event \(\{T \leq n\}\) or the event \(\{T > n\}\) in the role of the event \(\{T = n\}\) above).

• Recall that if \((M_n)_{n=0}^\infty\) is a martingale w.r.t. \((Y_n)_{n=0}^\infty\) and \(T\) is a stopping-time w.r.t. \((Y_n)_{n \geq 0}\), then \((M_n \wedge T)_{n=0}^\infty\) is also a martingale, where \(a \wedge b := \min(a, b)\). It is tempting to write
\[
\mathbb{E}[M_T] = \mathbb{E}[M_0], \text{ for all } n \geq 0.
\] (3)

• In general (3) may fail.

• Note that (3) is not the same as (2) since \(T\) is a stopping time and not a constant time.

**Theorem 0.0.1** (Optional Stopping Theorem). *If \((M_n)\) is a martingale w.r.t. \((Y_n)\) and \(T\) is a stopping time w.r.t. \((Y_n)\), then (3) holds if either of the following conditions hold

(i) \(|M_n \wedge T|\) is bounded and \(\Pr[T < \infty] = 1\).

(ii) \(\mathbb{E}[M_n^2]\) is bounded and \(\Pr[T < \infty] = 1\).

(iii) \(|\mathbb{E}[M_{n+1 \wedge T} - M_n \wedge T \mid Y_0, \ldots, Y_n]|\) is bounded and \(\mathbb{E}[T] < \infty\).

**Idea behind the proof:** Apply (2) to the martingale \(Q_n = M_n \wedge T\) at time \(n\). Take \(n\) to infinity. Verify that the error term \(\mathbb{E}[M_n 1_{T>n}]\) tends to 0 using one of the three conditions.

**Examples:**
1: Double or nothing. Let \( (X_n) \) be a sequence of 0-1 valued i.i.d. r.v.'s with \( \Pr[X_n = 1] = 1/2 = \Pr[X_n = 0] \). Let \( M_0 = 1 \) and \( M_{n+1} = 2M_nX_{n+1} \) for all \( n \). Then \((M_n)\) is a martingale w.r.t. \((X_n)\) and \( T = \inf\{t : M_t = 0\} \) is a stopping time w.r.t. \((X_n)\). However (3) fails

\[
\mathbb{E}[M_0] = 1 \neq 0 = \mathbb{E}[M_T].
\]

To see that \((M_n)\) is a martingale, note that \( M_n = 2^n \prod_{i=1}^n X_i \). Recall that if \( Y \) is determined by \( Z \), then \( \mathbb{E}[XY \mid Z] = Y \mathbb{E}[Y \mid Z] \) and that if \( X \) is independent of \( Z \), then \( \mathbb{E}[X \mid Z] = \mathbb{E}[X] \). Then

\[
\mathbb{E}[M_{n+1} \mid X_1, \ldots, X_n] = \mathbb{E}[2^{n+1} \prod_{i=1}^{n+1} X_i \mid X_1, \ldots, X_n] = 2^n \prod_{i=1}^n X_i \mathbb{E}[2X_{n+1} \mid X_1, \ldots, X_n]
\]

\[
= M_n \mathbb{E}[2X_{n+1}] = M_n \cdot 1 = M_n.
\]

2: Let \((S_n)\) be simple random walk on \( \mathbb{Z} \) with \( S_0 = 0 \). Let \( a > 0 \). Let \( T = \inf\{t : |S_n| = a\} \). Let \( M_n = S_n^2 - n \). Claim: \((M_n)\) is a martingale and \( T \) is a stopping time, both w.r.t. \((S_n)\). Before checking the martingale property (1) and verifying one of the three conditions of the Optional Stopping Theorem we show how this martingale, in conjunction with the optional stopping theorem, can be used to calculate \( \mathbb{E}[T] \):

\[
0 = M_0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[a^2 - T] = a^2 - \mathbb{E}[T],
\]

hence \( \mathbb{E}[T] = a^2 \).

Checking the martingale property (1): Write \( X_k = S_k - S_{k-1} \). Then \( X_1, X_2, \ldots \), are i.i.d. \( \pm 1 \)-valued r.v.'s with \( \Pr[X_k = 1] = 1/2 = \Pr[X_k = -1] \). Note that \( X_i^2 = 1 \) for all \( i \) and hence,

\[
S_{n+1}^2 - (n+1) = S_n^2 - n + 2X_{n+1}S_n + X_{n+1}^2 - 1 = S_n^2 - n + 2X_{n+1}S_n.
\]

Using the same properties of conditional expectation as in the previous example we get that

\[
\mathbb{E}[M_{n+1} \mid S_0, \ldots, S_n] = \mathbb{E}[S_{n+1}^2 - (n+1) \mid S_0, \ldots, S_n] = S_n^2 - n + 2S_n \mathbb{E}[X_{n+1}] = M_n + 0.
\]

Finally, to justify the application of the Optional Stopping Theorem we verify condition (iii). Note that for all \( i \in A_a := \{-a+1, -a+2, \ldots, a-2, a-1\} \) we have that \( \Pr[S_k+a \notin A_a \mid S_k = i] \geq 2^{-a} \), since the walk can make a consecutive steps in the same direction, away from \( A_a \). Hence (similarly to the proof of convergence to the stationary distribution using coupling you saw in class)

\[
\Pr[T > a(k+1) \mid T > ak] \leq 1 - 2^{-a} =: p.
\]

By induction, we get that

\[
\Pr[T > ak] \leq p^k.
\]

Whence

\[
\mathbb{E}[T] \leq a \sum_{k \geq 0} \Pr[T \geq ak] < \infty.
\]

Observe that on the event \( \{n \leq T\} \) we have that \( \mathbb{E}[M_{n+1} \wedge T - M_n \wedge T \mid Y_0, \ldots, Y_n] = 0 \) and on \( \{T > n\} \) we have that \( \mathbb{E}[M_{n+1} \wedge T - M_n \wedge T \mid Y_0, \ldots, Y_n] \leq 1 + \max_{i,j : |i|,|j| \leq 1} j^2 - i^2 \leq a^2 + 1. \)