Instructor: Allan Sly
Name: 
SID: 
Attempt all questions and show your working - solutions without explanation will not receive full credit. Answer the questions in the space provided. Additional space is available on the final page. Any notes you have written or typed yourself are permitted. Answers can be left in numerical form without simplification.

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Total
Question 1
Let $W(t)$ be a standard Brownian motion.
(a) [3 points] What is the distribution of $W_2 + W_4$?
(b) [4 points] Find Cov($W_2 + W_4, W_1 + W_3$).
(c) [4 points] Find $E(W_1 + W_3 | W_2 + W_4 = 1)$
Question 2

A computer game has 10 levels. If in one round you are at level $i$ you succeed with probability $0 < p_i < 1$ and move up to level $i + 1$ or fail with probability $1 - p_i$ and reattempt level $i$ in the next round. You win when you succeed in level 10 and after winning you go back to level 1 and start again. Let $X_n$ be the level you are at in round $n$.

(a) [4 points] Starting at level $i$ find the expected number of rounds until you win.
(b) [5 points] Find the stationary distribution of $X_n$
(c) [3 points] Over the long run what fraction of times do you win.
Question 3

A group of $N$ friends are each either Republican or Democrat. In each round, 2 randomly chosen friends are and they discuss politics, if they are from the same party they keep their original opinions but if they disagree one of them (at random) changes their opinion and favors the other party. Let $X_n$ be the number of Democrats after $n$ rounds.

(a) [3 points] Find the transition probabilities of $X_n$.

(b) [2 points] Is the Markov chain irreducible?

(c) [5 points] Given that $X_0 = i$, find the probability that eventually all the friends are Democrats.
Question 4

At a casino Bob the gambler begins with 1 dollar. In each round he bets all his money and it either doubles his money with probability $\frac{1}{2}$ or he loses it all with probability $\frac{1}{2}$. Let $X_n$ be his money after $n$ rounds.

(a) [3 points] Let $T$ be the first time he loses. Does the optional stopping theorem apply to $X$ and $T$?

(b) [5 points] In which of the following senses does $X_n$ converge, in probability, almost surely, in $L^1$ and in distribution.

(c) [Bonus 6 points] Suppose that the Markov chain is now modified so that after gambling Bob is given an additional dollar at the end of each round if he loses. In which of the following senses does $X_n$ converge, in probability, almost surely, in $L^1$ and in distribution.
Question 5

Let $Z_n$ be a branching process with $Z_0 = 0$ and offspring distribution $X = \text{Geom}(\frac{1}{3})$ (that is $\mathbb{P}[X = k] = \frac{1}{3^k}$).

(a) [4 points] Find $\mathbb{E}[Z_{10}]$.

(b) [5 points] Find $\mathbb{P}[Z_1 = 1 \mid Z_2 = 1]$.

(c) [4 points] Find the probability generating function of $X$.

(d) [4 points] Find the probability that the branching process survives for all time.
Question 6
Let $X(t)$ be a Markov chain with states $\{1, 2, 3, 4, 5, 6\}$. The transitions are as follows, when in state $i$ you wait time $Exp(i)$ and then roll a standard die and move to the value you rolled.

(a) [4 points] What is the generator of $X(t)$.
(b) [4 points] If $X(0) = 1$ what is the expected time until the first move to a different state.
(c) [5 points] What is the stationary distribution?
Question 7

A new university is opened at time 0. Students enrol at rate $\lambda$ and have age exponential with rate 1 (not very realistic).

(a) [3 points] What is the probability that the first student is older than the second student.

(b) [4 points] If 10 students enrolled by time 2, what is the conditional distribution of the number of students who enrolled by time 1.

(c) [5 points] What is the distribution of the number of students who enrolled at any time who were born before time 0.
Question 8
Let $Z_i$ be the population size of a Branching process with offspring distribution $X$ such that $X$ has mean 1 and variance $\sigma^2$. Suppose that the initial population is $Z_0 = n$.

[Bonus 10 points] Find $a_n$ and $b_n$ such that

$$\frac{1}{a_n} Z_{b_n t}$$

converges to a diffusion. Find the infinitesimal mean and variance of the diffusion.
Additional Space