There are 8 questions. Attempt all questions and show your working - solutions without explanation will **not** receive full credit. Answer the questions in the space provided. Additional space is available on the final page. Two double sided sheet of notes are permitted. Answers can be left in numerical form without simplification except where specified.
Question 1
Let $W(t)$ be a standard Brownian motion.
(a) [3 points] What is the distribution of $W_2 + W_4$?
(b) [4 points] Find $\text{Cov}(W_2 + W_4, W_1 + W_3)$.
(c) [4 points] Find $\mathbb{E}(W_1 + W_3 \mid W_2 + W_4 = 1)$

Solutions
(a) $W_2 + W_4 = 2W_2 + (W_4 - W_2)$. Since Brownian motion is a Gaussian process it is Gaussian and since the means are 0, its mean is 0.

$\text{Cov}(W_2 + W_4, W_2 + W_4) = \text{Cov}(W_2, W_2) + 2\text{Cov}(W_2, W_4) + \text{Cov}(W_4, W_4) = 2 + 2 \cdot 2 + 4 = 10$

Hence $W_4 + W_2 \sim N(0, 10)$.

(b) If $s \leq t$ then $W_s$ and $W_t - W_s$ are independent. Thus

$$\text{Cov}(W_s, W_t) = \text{Cov}(W_s, W_s) + \text{Cov}(W_s, W_t - W_s) = \text{Var}(W_s) + 0 = s.$$ 

Since $\text{Cov}$ is bilinear,

$$\text{Cov}(W_4 + W_2, W_1 + W_3) = \text{Cov}(W_1, W_2) + \text{Cov}(W_1, W_4) + \text{Cov}(W_2, W_3) + \text{Cov}(W_3, W_4)$$

$$= 2\text{Var}(W_1) + \text{Var}(W_2) + \text{Var}(W_3) = 2 + 2 + 3 = 7$$

(c) We can write $W_1 + W_3 = X + Y$, where $X := [(W_1 + W_3) - \frac{7}{10}(W_2 + W_4)]$ and $Y := \frac{7}{10}(W_2 + W_4)$. This decomposition was chosen so that

$$\text{Cov}(X, W_2 + W_4) = \text{Cov}(W_1 + W_3, W_2 + W_4) - \frac{7}{10}\text{Var}(W_2 + W_4) = 7 - 7 = 0.$$ 

Since we are in the setup of a Gaussian process, this implies that $X$ is independent of $W_2 + W_4$, and hence so of $Y$. Since the mean of $W_1 + W_3$ and of $Y$ is 0, this means that $X$ is a normal random variable with mean 0. If $W_2 + W_4 = 1$ then $Y = \frac{7}{10}$. Hence

$$\mathbb{E}[W_1 + W_3 \mid W_2 + W_4 = 1] = \mathbb{E}[X + Y \mid Y = 7/10] = \mathbb{E}[X] + 7/10 = 7/10.$$ 

Another way of getting to the same answer is obtained by considering the process

$$\tilde{B}_t := W_t - \frac{\text{Cov}(W_t, W_2 + W_4)}{\text{Var}(W_2 + W_4)}(W_2 + W_4).$$

From the RHS it is clear that $\text{Cov}(W_t, W_2 + W_4) = 0$ for all $0 \leq t \leq 4$. Similarly to the construction of the Brownian bridge, this makes the process $(\tilde{B}_t)_{0 \leq t \leq 4}$ independent of $W_2 + W_4$. As above, $\mathbb{E}[\tilde{B}_t] = 0$ for all $t \leq 4$, as a difference of two mean 0 Normal r.v.’s.

Denote $Z := W_2 + W_4$.

$$\mathbb{E}[X_1 + X_3 \mid Z = 1] = \mathbb{E}[\tilde{B}_1 + \tilde{B}_3 + \frac{\text{Cov}(W_1, Z) + \text{Cov}(W_3, Z)}{\text{Var}(Z)}Z \mid Z = 1]$$

$$= \mathbb{E}[\tilde{B}_1 \mid Z = 1] + \mathbb{E}[\tilde{B}_3 \mid Z = 1] + \mathbb{E}[\frac{\text{Cov}(W_1, Z) + \text{Cov}(W_3, Z)}{\text{Var}(Z)}Z \mid Z = 1]$$

$$= 0 + 0 + \frac{\text{Cov}(W_1 + W_3, W_2 + W_4)}{\text{Var}(W_2 + W_4)} \cdot 1 = \frac{7}{10}.$$
Question 2
A group of $N$ friends are each either Republican or Democrat. In each round, 2 randomly chosen friends are and they discuss politics, if they are from the same party they keep their original opinions but if they disagree one of them (at random) changes their opinion and favors the other party. Let $X_n$ be the number of Democrats after $n$ rounds.

(a) [3 points] Find the transition probabilities of $X_n$.

(b) [2 points] Is the Markov chain irreducible?

(c) [5 points] Given that $X_0 = i$, find the probability that eventually all the friends are Democrats.

Solutions

(a and b) 0 and $N$ are absorbing states. In particular, the chain is reducible (not irreducible, since from 0 the chain can’t get to any other state).

For all $1 < i < N$, the chain can move from $i$ only to $i \pm 1$ or else it stays at $i$. The probability of moving from $i$ to $i + 1$ is equal to the probability of moving to $i - 1$. Namely, this probability is half the probability of picking one Republican and one Democrat, given that there are $i$ and $N - i$ of each. The probability of staying at $i$ is simply the probability of picking two people from the same group. Hence for all $1 \leq i < N$

$$P(i, i) = \frac{i(i - 1) + (N - i)(N - i - 1)}{N(N - 1)},$$

$$P(i, i + 1) = \frac{i(N - i)}{N(N - 1)} = P(i, i - 1).$$

(c) Note that $X_n$ is a martingale. Then if $T$ is the hitting time of an absorbing state,

$$i = \mathbb{E}[X_T \mid X_0 = i] = N \mathbb{P}[X_T = N \mid X_0 = i]$$

so the probability is $i/N$.

Alternatively observe that whenever the chain is at some $0 < j < N$ its next step away from $j$ is to $j \pm 1$ with equal probability. Thus the hitting probabilities of the end-points 0, $N$ are the same as for SRW on the interval $\{0, 1, \ldots, N\}$. Thus $\Pr[T_N < T_0 \mid X_0 = i] = \frac{i}{N}$ (where $T_j := \inf\{t : X_t = j\}$).
Question 3
A computer game has 10 levels. If in one round you are at level \(i\) you succeed with probability \(0 < p_i < 1\) and move up to level \(i + 1\) or fail with probability \(1 - p_i\) and reattempt level \(i\) in the next round. You win when you succeed in level 10 and after winning you go back to level 1 and start again. Let \(X_n\) be the level you are at in round \(n\).

(a) [4 points] Starting at level \(i\) find the expected number of rounds until you win.
(b) [5 points] Find the stationary distribution of \(X_n\)
(c) [3 points] Over the long run what fraction of times do you win.

Solutions
Let \(Y_i\) be the number of attempts until step \(i\) is passed successfully (counting from the first time we reach level \(i\) until it is passed). That is, \(Y_i = T_i - T_{i+1}\), with addition modulo 10. Then \(Y_1, \ldots, Y_{10}\) are independent Geometric r.v.'s with \(Y_i \sim \text{Geom}(p_i)\).

Starting from level \(i\) the time needed to win is \(\sum_{j=i}^{10} Y_j\). So its mean is

\[
\sum_{j=i}^{10} \mathbb{E}[Y_j] = \sum_{j=i}^{10} \frac{1}{p_j}.
\]

(b)

Solution 1: The definition of the stationary distribution is that \(\pi P = \pi\) and so

\[
\pi_i = p_{i-1} \pi_{i-1} + (1 - p_i) \pi_i
\]

and so \(\pi_i = \frac{p_{i-1}}{p_i} \pi_{i-1}\). Iterating this argument we have that

\[
\pi_i = \pi_0 \prod_{j=1}^{i} \frac{p_{j-1}}{p_j} = \frac{\pi_0 p_0}{\pi_i}
\]

Since \(\sum_i \pi_i = 1\) we have

Solution 2: Intuitively the fraction of time spent in state \(i\) should be proportional to \(p_i^{-1}\), which implies that \(\pi_i = \frac{p_i^{-1}}{\sum_{j=1}^{10} p_j}\). Viewing this as a guess and then check that

\[
\pi_i P = \pi_i P(i, i) + \pi_{i-1} P(i - 1, i) = \frac{p_i^{-1}(1 - p_i)}{\sum_{j=1}^{10} p_j^{-1}} + \frac{p_{i-1}^{-1} p_{i-1}}{\sum_{j=1}^{10} p_j^{-1} p_{i-1}} = \frac{p_i^{-1}}{\sum_{j=1}^{10} p_j^{-1}} = \pi_i,
\]

(with \(i - 1\) interpreted modulo 10).

Another way of verifying this:
This is indeed the case since for large \(t\), the number of times by time \(t\) in which the chain enters state \(i\) from a different state (from \(i - 1\) modulo 10), equals (up to an error of at most 2) to the number of times the chain made the cycle from level 1 back to level 1 from level 10 (after winning). By the Law of Large Numbers this occurs roughly \(\frac{t}{\sum_{j=1}^{10} p_j}\) times. Every time the chain enters \(i\) (from \(i - 1\)) it stays there for \(p_i^{-1}\) time units at average. So by the Law of Large Numbers, the asymptotic fraction of time spent at level \(i\) is indeed \(\frac{p_i^{-1}}{\sum_{j=1}^{10} p_j}\).

(c) The asymptotic frequency of transitions from 10 to 1 is \(\pi_{10} P(10, 1) = \frac{1}{\sum_{j=1}^{10} p_j}\).
Question 4
Let $Z_n$ be a branching process with $Z_0 = 0$ and offspring distribution $X = \text{Geom}(\frac{1}{3})$ (that is $P[X = k] = \frac{2^k}{3^k}$).
(a) [4 points] Find $E[Z_{10}]$.
(b) [5 points] Find $P[Z_1 = 1 \mid Z_2 = 1]$.
(c) [4 points] Find the probability generating function of $X$.
(d) [4 points] Find the probability that the branching process survives for all time.

Solutions
(a) The mean offspring distribution is $\mu = 2$. Thus $E[Z_{10}] = 2^{10}$.
(b) $P[Z_1 = 1 = Z_2] = P[X = 1]^2 = \frac{4}{81}$.
For any $k \geq 1$,

$P[Z_2 = 1 \mid Z_1 = k] = \binom{k}{1} P[X = 1](P[X = 0])^{k-1} = k \times \left(\frac{2}{9}\right) \times \left(\frac{1}{3}\right)^{k-1}$.

By the total probability formula

$P[Z_2 = 1] = \sum_{k \geq 1} P[Z_1 = k] P[Z_2 = 1 \mid Z_1 = k] = \sum_{k \geq 1} \frac{2^k}{3^{k+1}} \times \frac{2k}{3^{k+1}} = \frac{4}{63} \sum_{k \geq 1} \left(\frac{2}{9}\right)^{k-1} \frac{7}{9} k$

$= \frac{4}{63} \times \frac{9}{7} = \frac{4}{49}$.

So by Bayes’ rule

$P[Z_1 = 1 \mid Z_2 = 1] = \frac{P[Z_1 = 1 = Z_2]}{P[Z_2 = 1]} = \frac{4/81}{4/49} = 49/81$.

(c) $G_X(s) = \sum_{k \geq 0} P[X = k] s^k = \frac{1}{3} \sum_{k \geq 0} \left(\frac{2s}{3}\right)^k = \frac{1}{3 \left(1 - \frac{2s}{3}\right)} = \frac{1}{3 - 2s}$.

(d) Since $\mu = 2 > 1$, $P[\text{the process survives forever}] = 1 - \rho$, where $\rho = G_X(\rho)$ and $0 < \rho < 1$. Solving this equation yields

$3\rho - 2\rho^2 = 1 \iff (2\rho - 1)(\rho - 1) = 0 \iff \rho = 1/2$. 
Question 5
Kangaroos hop by your office at rate 1 Poisson process, each one is independently red with probability $2/3$ and grey with probability $1/3$. Let $X(t)$ be the indicator that the most recent kangaroo is grey.

(a) [5 points] Find the generator of $X(t)$.
(b) [4 points] Find the transition probability $P[X(t) = 1 | X(0) = 1]$.
(c) [3 points] What is the stationary distribution of $X(t)$.
(d) [4 points] Suppose that $X(t)$ is started from its stationary distribution and let $T$ be the first time that $X$ changes state. Find the density of $T$.

Solutions
When the chain is at state 0 (resp. 1) it moves to state 1 (resp. 0) after a random time $\sim \text{Exp}(2/3)$ (resp. $\text{Exp}(1/3)$). Hence

$$G = \begin{pmatrix} -1/3 & 1/3 \\ 2/3 & -2/3 \end{pmatrix}$$

(b) After the arrival of the first Kangaroo the distribution of the chain is $(2/3, 1/3)$ and it continues to have that distribution for all larger times. In particular $\pi_0 = 2/3$ and $\pi_1 = 1/3$. Indeed this vector satisfies the detailed balance equation $-1/3 \pi_0 + 2/3 \pi_1 = 0$.

Moreover, if $T$ is the first arrival time of a Kangaroo, it follows that

$$\Pr[X(t) = 1 | X(0) = 1] = \Pr[T > t] + \frac{1}{3} \Pr[T \leq t] = \frac{1}{3} + \frac{2}{3} \Pr[T > t] = \frac{1}{3}(1 + 2e^{-t}).$$

Second way: we always have for an irreducible reversible chain with a state space of size $k$ (any chain on 2 states which does not have an absorbing state is a birth and death chain and thus reversible) that $\Pr[X(t) = y | X(0) = x] = \pi(y) + \sum_{j=2}^{k} c_j(x, y)e^{-\lambda_j t}$, where $\lambda_2, \ldots, \lambda_k$ are the non-zero eigenvalues of the generator $G$. For $k = 2$ this takes a particularly simple form. We can find $\pi$ using the detailed balance equation. Then all that is left is to find $\lambda_2$. The constants $c_2(x, y)$ must be the ones which make $\Pr[X(0) = y | X(0) = x] = 1_{x=y}$ (after $\pi$ was calculated there is a unique solution). In this case, $\lambda_2 = 1$ and this leads to the same answer as before.

Third way (uses implicitly some of the reasoning of the 2nd way): by the forward equation

$$\frac{d}{dt} \Pr[X(t) = 1 | X(0) = 1] = G_{0,1} \Pr[X(t) = 0 | X(0) = 1] + G_{1,1} \Pr[X(t) = 1 | X(0) = 1]$$

$$= \frac{1}{3}(1 - \Pr[X(t) = 1 | X(0) = 1]) - \frac{2}{3} \Pr[X(t) = 1 | X(0) = 1]$$

$$= \frac{1}{3} - \Pr[X(t) = 1 | X(0) = 1].$$

Since $\pi_1 = 1/3$ we got that

$$\frac{d}{dt} [\Pr[X(t) = 1 | X(0) = 1] - \pi_1] = -[\Pr[X(t) = 1 | X(0) = 1] - \pi_1].$$
Solving this gives $\Pr[X(t) = 1 \mid X(0) = 1] - \pi_1 = Ce^{-t} \implies \Pr[X(t) = 1 \mid X(0) = 1] = \frac{1}{3} + \frac{2e^{-t}}{3}$. The constant $C$ has to be $2/3$ in order to ensure that for $t = 0$ we get 1.

(c) As we showed in part (b), $\pi = (\frac{2}{3}, \frac{1}{3})$.

(d) w.p. $1/3$, $T \sim \text{Exp}(2/3)$ and w.p. $2/3$, $T \sim \text{Exp}(1/3)$. Hence

$$f_T(t) = \frac{1}{3}f_{\text{Exp}(2/3)}(t) + \frac{2}{3}f_{\text{Exp}(1/3)}(t) = \frac{1}{3}(2/3)e^{-2t/3} + \frac{2}{3}(1/3)e^{-t/3} = (2/9)[e^{-t/3} + e^{-2t/3}].$$
Question 6

Cars arrive according to a Poisson process with rate $\lambda(t) = 1 + t$.

(a) [4 points] Let $T_1$ be the time of the first arrival, find the distribution of $T_1$.

(b) [3 points] Let $T_2$ be the arrival time of the second car. Is $T_2 - T_1$ independent of $T_1$?

(c) [5 points] Given that 2 cars arrived by time 1 find the conditional distribution of $T_1$.

Solutions

(a) Let $N_t$ be the number of points in $[0,t]$. Then $N_t$ is a Poisson r.v. with mean $\int_0^t \lambda(s)ds = t + t^2/2$. Thus

$$\Pr[T_1 > t] = \Pr[N_t = 0] = e^{-(t+t^2/2)}.$$

(b) They are not independent, since $\lambda(t)$ is not fixed. In fact, since it increases in $t$, it is easy to see, using similar reasoning as part (a), that if $t_1 < t_2$, then the conditional distribution of $T_2 - T_1$, given that $T_1 = t_i$ satisfies $\Pr[T_2 - T_1 > t \mid T_1 = t_1] > \Pr[T_2 - T_1 > t \mid T_1 = t_2]$ for all $t > 0$.

(c) Let $N(A)$ be the number of points in the set $A$. We may view them as un-ordered sequence of points. Namely, if the points are ordered according to the time of arrival, we can take a random perturbation of their indices.

A general fact is that given that there are $k$ points in a set $A$, their joint distribution (as un-ordered points) is that of $k$ i.i.d.’s whose density is given by $f(x) := \frac{\lambda(x)1_{x \in A}}{\int_A \lambda(s)ds}$.

In our setup $A = [0,1]$, $k = 2$ and $f(x) = \frac{2(1+t)}{3}$. Denote the two un-ordered points by $X_1, X_2$. Then (given that $N_1 = 2$) $T_1 = \min(X_1, X_2)$ and by independence

$$\Pr[T_1 \geq t \mid N_1 = 2] = \Pr[\min(X_1, X_2) \geq t] = \Pr[X_1 \geq t \text{ and } X_2 \geq t] = \Pr[X_1 \geq t] \Pr[X_2 \geq t] = \Pr[X_1 \geq t]^2.$$

Now

$$\Pr[X_1 \geq t] = \frac{2}{3} \int_t^1 (1+s)ds = \frac{2}{3} (\frac{3}{2} - t - t^2/2)$$

and so

$$\Pr[T_1 \leq t \mid N_1 = 2] = 1 - \Pr[X_1 \geq t]^2 = 1 - \left(\frac{2}{3} \left(\frac{3}{2} - t - t^2/2\right)\right)^2.$$
Question 7

Buses arrive with gaps given by independent random variables with density \( f(x) = x/50 \) for \( x \in [0, 10] \). Let \( N_t \) be the number of arrivals by time \( t \).

(a) [3 points] Find \( \lim_{t \to \infty} \frac{1}{t} N_t \).

(b) [4 points] You arrive at a random point in time late in the day, find the expected time to wait until the next bus.

(c) [5 points Bonus] You have waited 5 minutes for the bus so far. Conditional on that find the expected amount of time you further wait.

Solutions

(a) By the renewal theorem \( \lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{10} \int_0^{10} x f(x) dx \) = \( \frac{15}{100} \), where the denominator \( \int_0^{10} x f(x) dx \) represents the expectation of the spacing between each two buses.

(b) The distribution of the time until the next bus arrives tends to \( U \cdot W \), where \( U \sim U[0, 1] \) and \( W \) is the sized biased version of the distribution of the spacing between two buses, and \( W \) and \( U \) are independent. The density of \( W \) is \( f_W(x) = \frac{1}{3} \int_0^{10} y f(y) dy \) = \( \frac{3}{100} \) for \( 0 \leq x \leq 10 \). Hence the expectation is

\[
E[UW] = E[U]E[W] = \frac{1}{2} \left( \int_0^1 y f_W(y) dy \right) = \frac{3}{8} \cdot \frac{3}{100} = \frac{9}{400}.
\]

(c) If \( X \) is the original distribution or the gaps, the density of the waiting time \( V \) can also be given as

\[
f_V(v) = \frac{P[X \geq v]}{E[X]} = \frac{\int_v^{10} x/50 dx}{20/3} = \frac{3(100 - v^2)}{2000}
\]

for \( 0 \leq v \leq 10 \). Then the further time to wait given that you have waited 5 minutes already is

\[
E[V-5 \mid V \geq 5] = \frac{E[(V-5)I(V \geq 5)]}{P[V \geq 5]} = \frac{\int_5^{10} (v-5)f_V(v) dv}{\int_5^{10} f_V(v) dv} = \frac{\int_5^{10} (v-5) \frac{3(100-v^2)}{2000} dv}{\int_5^{10} \frac{3(100-v^2)}{2000} dv} = \frac{35/64}{5/16} = \frac{7}{4}
\]
Question 8
Let $W_t$ be standard Brownian motion.
(a) [5 points] Find the joint distribution of $W_{1/3}, W_{2/3}$ conditional on $W_1 = 1$.
(b) [5 points] In terms of $\Phi$ the CDF of the standard normal find

$$\mathbb{P}[W_{1/3} > W_{2/3} \mid W_1 = 1].$$

Solutions
(a) As we have shown many times before, for $t \in [0, 1]$ we have that $\mathbb{E}[W_t \mid W_1] = tW_1$. Then $W_t - \mathbb{E}[W_t \mid W_1] = W_t - tW_1$ is independent of $W_1$. So after we condition on $W_1 = 1$ we get that

$$W_{1/3} = (W_{1/3} - \frac{1}{3}W_1) + \frac{1}{3}W_1 = (W_{1/3} - \frac{1}{3}W_1) + \frac{1}{3},$$

$$W_{2/3} = (W_{2/3} - \frac{2}{3}W_1) + \frac{2}{3}W_1 = (W_{2/3} - \frac{2}{3}W_1) + \frac{2}{3}.$$  

Thus $\mathbb{E}[(W_{1/3}, W_{2/3}) \mid W_1 = 1] = (1/3, 2/3)$.

By independence

$$\text{Var}(W_{1/3} \mid W_1 = 1) = \text{Var}(W_{1/3} - W_1/3) + \text{Var}(W_{1/3}) = \text{Var}(W_{1/3} - W_1/3) = \text{Var}(W_{1/3}) + \text{Var}(W_1) / 3.$$  

Similarly

$$\text{Var}(W_{1/3} \mid W_1 = 1) = \text{Var}(W_{2/3} - W_1/3) = 2/9.$$  

Using the fact that covariance is bilinear and that $\text{Cov}(X + \text{constant}, Y) = \text{Cov}(X, Y)$ (together with $\text{Cov}(W_s, W_t) = \text{min}(s, t)$), their covariance is

$$\text{Cov}(W_{1/3} - W_{1/3}, W_{2/3} - 2W_{1/3}) = \text{Cov}(W_{1/3}, W_{2/3}) - \frac{2}{3} \text{Cov}(W_{1/3}, W_1)$$

$$- \frac{1}{3} \text{Cov}(W_1, W_{2/3}) + \frac{2}{9} \text{Var}(W_1) = 1/3 - (2/3)(1/3) - (1/3)(2/3) + 2/9 = 1/9.$$  

So their (conditional) joint distribution is that of bivariate normal with mean vector $(1/3, 2/3)$, and Covariance matrix

$$\begin{pmatrix}
(1/3)^2 \cdot 2 & (1/3)(2/3) \min(2, 1/2) \\
(1/3)(2/3) \min(2, 1/2) & (2/3)^2(1/2)
\end{pmatrix} = \begin{pmatrix}
2/9 & 1/9 \\
1/9 & 2/9
\end{pmatrix}.$$  

Alternative - using time reversal: $B_t := tW_{1/t}$ is a BM. Also, $\bar{B}_t = B_{t+1} - B_1$ is a BM independent of $B_1$.

$$W_1 = 1 \rightarrow B_1 = 1$$

$$W_{1/3} \rightarrow \frac{1}{3}B_3 = \frac{1}{3}[(B_3 - B_1) + B_1] = \frac{1}{3}[B_2 + 1]$$

$$W_{2/3} \rightarrow \frac{2}{3}B_{3/2} = \frac{2}{3}[(B_{3/2} - B_1) + B_1] = \frac{2}{3}[\bar{B}_{1/2} + 1].$$
So their (conditional) joint distribution is that of bivariate normal with mean vector \((\frac{1}{3}, \frac{2}{3})\),
and Covariance matrix
\[
\begin{pmatrix}
\frac{1}{3} \cdot 2 & \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{3}{2} \\
\frac{3}{3} \cdot \frac{3}{2} \cdot \frac{3}{2} & \frac{3}{3}^2 \cdot \frac{3}{2}
\end{pmatrix} = \begin{pmatrix}
\frac{2}{9} & \frac{1}{9} \\
\frac{1}{9} & \frac{2}{9}
\end{pmatrix}.
\]

(b) This is now a question of a pair of bi-variate normals. Let \(X, Y\) have a bivariate normal
distribution with mean vector \((1/3, 2/3)\), and covariance matrix \(\begin{pmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{pmatrix}\). Then we will
calculate \(\Pr[X > Y] = \mathbb{P}[X - Y > 0]\). Now \(X - Y\) is normal and has mean \(-\frac{1}{3}\) and

\[
Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y) = \frac{2}{9}
\]

so the distribution of \(X - Y\) is \(N(-\frac{1}{3}, \frac{2}{9})\). Thus

\[
\mathbb{P}[X > Y] = \mathbb{P}[N(-\frac{1}{3}, \frac{2}{9}) > 0] = \mathbb{P}[N(0, 1) > 1/\sqrt{2}] = 1 - \Phi(1/\sqrt{2}).
\]
Additional Space