Let $\mathbb{Z}$ be the set of all integers \{0, 1, -1, 2, -2, \ldots\} and let $\mathbb{N}$ be the positive integers \{1, 2, 3, \ldots\}. Denote the rational numbers by $\mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0 \}$. The ancient Greeks already discovered that rational numbers are not sufficient to describe certain natural geometrical quantities, such as the diagonal in a square of side 1.

**Proposition 0.1.** $\sqrt{2} \notin \mathbb{Q}$. That is, for every $a, b \in \mathbb{Z}$ with $b \neq 0$, we have $(a/b)^2 \neq 2$.

**Proof.** Suppose $(a/b)^2 = 2$ with $a, b \in \mathbb{Z}$. We may assume that $a, b > 0$, otherwise we replace $a, b$ by their absolute values. We also may assume that we chose a solution with $a$ minimal.

The equation $a^2 = 2b^2$ implies that $a$ is even, and therefore $a^2$ is divisible by 4. Consequently $b^2 = a^2/2$ is even whence $b$ is even. Therefore we can replace $a$ and $b$ by $a/2$ and $b/2$, and obtain a smaller pair of integers where the ratio of their squares is 2. This contradicts the minimality of $a$. \hfill \Box

The construction of the real numbers $\mathbb{R}$ can be done either via Dedekind cuts, or using Cauchy sequences. A Dedekind cut $A|B$ consists of a pair of disjoint nonempty sets $A, B \subset \mathbb{Q}$, such that $A \cup B = \mathbb{Q}$ and $a < b$ holds for all $a \in A$ and $b \in B$. We also require that $A$ has no largest element.

A pertinent example of a Dedekind cut is $A|B$ where

(1) \quad $A = \{ x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2 \}$ and $B = \{ x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2 \}$.

We will return to Dedekind cuts later.

Recall that a sequence $\{x_n\}$ converges to a limit $L$ (in symbols, $x_n \to L$ as $n \to \infty$) if for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > n_0$. For now, focus on $x_n, L, \epsilon \in \mathbb{Q}$. This also applies to the next definition. However, these definitions will apply more generally later. We need a more sophisticated definition that describes when the members of a sequence are getting closer to each other without referring to any limit.

**Definition 0.2.** A sequence $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence if for all (rational) $\epsilon > 0$, there exists an $N$ such that $m, n > N \Rightarrow |x_m - x_n| < \epsilon$.

For example, the sequence $\{3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots\}$ where each time we add another digit in the decimal expansion of $\pi$, is a Cauchy sequence. As we shall see later in the course, $\pi \notin \mathbb{Q}$, so this sequence does not converge in $\mathbb{Q}$. Similarly, if $x_n^2 \to 2$, then $\{x_n\}$ cannot converge to any rational $L$.

**Problem 0.3** (Challenge). Find an explicit sequence $\{x_n\} \subset \mathbb{Q}$ such that $x_n^2 \to 2$ for all $x_n > 0$.

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Following the preceding example, we can take \( x_1 = 1.4 \), and \( x_n = x_{n-1} + \frac{a_n}{10^n} \) for \( n > 1 \), where \( a_n \) is the largest integer \( a \) such that \((x_{n-1} + \frac{a_n}{10^n})^2 < 2\). Then \( \{x_n\} \) is a Cauchy sequence, and \( x_n^2 \to 2 \) as \( n \to \infty \).

Here is an idea for a more insightful solution, motivated by a standard algorithm to approximate square roots. Let

\[
\begin{align*}
(2) & \quad x_1 = 2 \quad \text{and} \quad x_n = \frac{1}{2} \left( x_{n-1} + \frac{2}{x_{n-1}} \right) \quad \text{for} \quad n > 1 .
\end{align*}
\]

By induction, \( x_n \in \mathbb{Q} \) for all \( n \).

**Problem 0.4** (Exercise). For the sequence in (2), check that \( x_{n+1} < x_n \) for all \( n > 0 \) and that the Cauchy property holds. Hint: Consider \( x_n^2 - 2 \).

To ensure that a sequence \( \{y_n\} \) is Cauchy, it is not enough to verify that \( y_n - y_{n-1} \to 0 \) as \( n \to \infty \).

**Example 0.5.** Consider \( H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \), so that \( H_n - H_{n-1} = \frac{1}{n} \to 0 \). Nevertheless, \( \{H_n\} \) is not a Cauchy sequence. To see this, take \( \epsilon = 1/3 \), for instance. Given any \( N \), we must find \( m, n > N \) with \( |H_n - H_m| \geq 1/3 \). Let \( m = N + 1 \) and \( n = 2m \). Then

\[
H_{2m} - H_m = \frac{1}{m + 1} + \frac{1}{m + 2} + \cdots + \frac{1}{2m} \geq \frac{m}{2m} = \frac{1}{2} .
\]

We are done.

In the preceding example, the sequence \( H_n \) is not bounded.

**Problem 0.6** (Exercise).  

- Show that every Cauchy sequence is bounded.
- Show that every convergent sequence is a Cauchy sequence.
- Find an example of a bounded sequence \( \{y_n\} \) such that \( y_n - y_{n-1} \to 0 \) yet \( \{y_n\} \) is not a Cauchy sequence. Hint: Consider the distance from \( H_n \) to the nearest integer.

To define real numbers via Cauchy sequences, we must deal with the fact that many different sequences might converge to the same limit.

**Definition 0.7.** Suppose \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences of rational numbers. We say that \( \{x_n\} \) is **equivalent to** \( \{y_n\} \), and write \( \{x_n\} \sim \{y_n\} \), if \( x_n - y_n \to 0 \).

Given a Cauchy sequence \( \{x_n\} \subset \mathbb{Q} \), consider its **equivalence class**

\[
\overline{\{x_n\}} = \{ \text{all sequences} \ \{y_n\} \ \text{such that} \ \{x_n\} \sim \{y_n\} \}.
\]

We can define a real number as such an equivalence class. To do so, and still think of \( \mathbb{Q} \) as a subset of \( \mathbb{R} \), we identify every rational number with the equivalence class of (Cauchy) sequences converging to it.

University of California, Berkeley  
E-mail address: peres@stat.berkeley.edu