1 Theoretical Problems

Consider the time series

\[ x_t = \beta_1 + \beta_2 t + Z_t \quad (t = 1, 2, 3, \ldots) \tag{1} \]

where \( \beta_1 \) and \( \beta_2 \) are known constants and \( Z_t \) is a white noise process with variance \( \sigma_Z^2 \).

1. Determine whether \( x_t \) is stationary.

\[ E(x_t) = E(\beta_1 + \beta_2 t + Z_t) = \beta_1 + \beta_2 t \quad \text{using} \quad E(Z_t) = 0 \]

As the mean function depends on time and is not constant, \( x_t \) is non-stationary.

2. Show that the process \( y_t = x_t - x_{t-1} \) is stationary.

We need to check the three requirements of a stationary process given in the handouts.

\[ E(y_t) = E(x_t - x_{t-1}) = E(\beta_1 + \beta_2 t + Z_t - (\beta_1 + \beta_2 (t-1) + Z_{t-1})) = \beta_2 \]

where we used that \( E(Z_t) = 0 \) \( \forall t \). So the mean function is constant. Also, \( y_t \) has a finite second moment since it is a sum of two random variables with finite second moments. The only thing left to verify is that the auto covariance function is only a function of the lag \( h \), and does not depend on time \( t \).

\[ \gamma(t, t+h) = Cov(y_t, y_{t+h}) = E[y_t y_{t+h}] - E[y_t]E[y_{t+h}] \tag{2} \]

Working with the first part of (2) we have the following result:

\[ E[y_t y_{t+h}] = E[(x_t - x_{t-1})(x_{t+h} - x_{t+h-1})] \]
\[ = E[(Z_t + \beta_2 - Z_{t-1})(Z_{t+h} + \beta_2 - Z_{t+h-1})] \]
\[ = E[Z_t Z_{t+h}] - E[Z_t Z_{t+h-1}] + \beta_2^2 + E[Z_{t-1} Z_{t+h-1}] \]

where we used that \( E(Z_1 Z_{t_2}) = 0 \ \forall t_1 \neq t_2 \). Also using \( E(Z_t^2) = \sigma_Z^2 \) gives:
3. Show that the mean of the moving average
\[ v_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j} \]  
(3)
is \( \beta_1 + \beta_2 t \), and give a simplified expression for the autocovariance function of \( v_t \).

\[
E(v_t) = \frac{1}{2q+1} \sum_{j=-q}^{q} E(x_{t-j}) = \frac{1}{2q+1} \sum_{j=-q}^{q} E(\beta_1 + \beta_2(t-j) + Z_{t-j}) = \beta_1 + \beta_2 \frac{t}{2q+1} \sum_{j=-q}^{q} (t-j) = \beta_1 + \beta_2 t + \frac{\beta_2}{2q+1} \sum_{j=-q}^{q} j = \beta_1 + \beta_2 t
\]

For the autocovariance function it is useful to first calculate:

\[
E(x_t x_{t-j}) = E[(\beta_1 + \beta_2 t + Z_t)(\beta_1 + \beta_2(t-j) + Z_{t-j})] \begin{cases} 
\beta_1^2 + 2\beta_1\beta_2 t + \beta_2^2 t^2 + \sigma_Z^2 & \text{if } j = 0 \\
\beta_1^2 + 2\beta_1\beta_2 t + \beta_1\beta_2 j + \beta_2^2 t(t-j) & \text{else}
\end{cases}
\]

Using this, we can calculate:

\[
E[v_t v_{t+h}] = \frac{1}{(2q+1)^2} E[\sum_{j=-q}^{q} x_{t-j} \sum_{i=-q}^{q} x_{t+h-i}] = \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{i=-q}^{q} E[x_{t-j} x_{t+h-i}] = \beta_1^2 + 2\beta_1\beta_2 \frac{(2q+1)^2}{2q+1} \sum_{j=-q}^{q} \sum_{i=-q}^{q} (t-j) + \beta_1\beta_2 \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{i=-q}^{q} (h+j-i) + ... + \beta_2^2 \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{i=-q}^{q} [(t-j)(t-j) + (h+j-i)] + \sigma_Z^2 \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{i=-q}^{q} 1_{h+j-i=0} = \beta_1^2 + 2\beta_1\beta_2 t + \beta_1\beta_2 h + \beta_2^2 t(t+h) + \sigma_Z^2 \frac{1}{(2q+1)^2} \sum_{j=-q}^{q} \sum_{i=-q}^{q} 1_{h+j-i=0}
\]
Also:

\[
E[v_t|E[v_{t+h} = (\beta_1 + \beta_2 t)(\beta_1 + \beta_2 (t + h))] = \beta_1^2 + 2\beta_1\beta_2 t + \beta_1\beta_2 h + \beta_2^2 t(t + h)
\]

Using above results, we can simplify the expression for the autocovariance function of \( v_t \) as:

\[
\text{Cov}(v_t, v_{t+h}) = E[v_t v_{t+h}] - E[v_t]E[v_{t+h}] = \sigma^2 Z \left( \frac{1}{2q + 1} \right)^2 \sum_{j=-q}^{q} \sum_{i=-q}^{q} 1_{h+j-i=0} = \left\{ \begin{array}{ll}
\frac{\sigma^2 Z \left( \frac{2q+1-h}{2q+1} \right)}{2q+1} & \text{if } |h| \leq 2q \\
0 & \text{else}
\end{array} \right.
\]