Homework 2.

1. Let \{X_t\} be a stationary time series with mean \(\mu_X\) and autocovariance function \(\gamma_X(h)\). Consider estimating \(\mu_X\) by

\[
\bar{X}_n = \frac{1}{n} \sum_{t=1}^{n} X_t
\]

the sample mean over time.

(a) Prove that this estimate is unbiased, that is,

\[
E\{\bar{X}_n\} = \mu_X
\]

(b). Prove that the variance of the sample mean can be written as

\[
E\{(\bar{X}_n - \mu)^2\} = \frac{1}{n} \gamma_X(0) + \frac{2}{n} \sum_{h=1}^{n-1} (1 - \frac{h}{n}) \gamma_X(h)
\]

(c) To what does the variance reduce in (b) when \(\{X_t\}\) is a white noise process with variance \(\sigma^2\). Comment on the result.

2. A concept which is useful in geostatistics is that of the variogram, defined for a stationary process, \(\{X_t\}\), as

\[
\frac{1}{2} E\{(X_{t+d} - X_t)^2\}
\]

(a) Show that the variogram can be expressed in the form

\[
\gamma_X(0) - \gamma_X(d)
\]

where \(\gamma_X(h)\) is the usual lag-\(h\) autocovariance function.

(b) Use this result to show that

\[
|\gamma_X(h)| \leq \gamma_X(0), \quad h = 0, \pm 1, \pm 2, ...
\]

for \(\{X_t\}\) a stationary time series.