Definitions.

A quadratic form is an expression of the form $y^T Ay$ where $y$ and $A$ are $n \times 1$ and $n \times n$ respectively. Its rank is $r(A)$.

Let the $Y_i$ be $IN(\mu_i,1)$ for $i = 1,\ldots,n$. Let $Y = [Y_i]$ and $\mu = [\mu_i]$ be $n \times 1$. The non-central chi-square, $\chi^2(n,\lambda)$, is the distribution of $Y^T Y$ with $\lambda = \mu^T \mu$. Its expected value is $n + \lambda$.

Cochran’s Theorem. Let the $Y_i$ be $IN(\mu_i,\sigma^2)$ for $i = 1,\ldots,n$. Let $Q_1,\ldots,Q_J$ be quadratic forms with ranks $n_1,\ldots,n_J$, such that

$$Y^T Y = Q_1 + \ldots + Q_J$$

Then a necessary and sufficient condition that $Q_j$ is $\sigma^2 \chi^2(r_j,\lambda_j)$ and the quadratic forms are independent is $n = r_1 + \ldots + r_J$ in which case $\lambda_j = \mu^T A_j \mu$ if $Q_j = Y^T A_j Y$ and $\sum \mu_j^2 = \sum \lambda_j$.

Example. (See Handout 1.)

Suppose that $Y = X\beta + W$ with $W$ being $N_n(0,\sigma^2 I_n)$ and $X n \times p$. Then $Y^T Y$ is $\sigma^2 \chi^2(n, \left| X\beta \right|^2/{\sigma^2})$.

Suppose $X$ is of full rank $p < n$. Then

$$\left| Y - X\hat{\beta} \right|^2$$

is $\sigma^2 \chi^2(n-p,0)$

Suppose that $H_0$ is the hypothesis that $P^T \beta = 0$ where $P$ is $p \times s$ and of full rank $s < p$. Then under $H_0$

$$\hat{\beta}_* = \hat{\beta} - (X^T X)^{-1} P [P^T (X^T X)^{-1} P]^{-1} P^T \hat{\beta},$$

$$\left| X(\hat{\beta} - \hat{\beta}_*)^2 \right|$$

is $\sigma^2 \chi^2(s,0)$
and

\[ |X \hat{\beta}_*|^2 \sim \sigma^2 \chi^2(p - s), \quad |X \beta_*|^2 \sigma^2 \]

**References.** See those of Handout 1.