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1. Review: Causality, invertibility, AR(p) models
2. ARMA(p,q) models
3. Stationarity, causality and invertibility
4. The linear process representation of ARMA processes: $\psi$.
5. Autocovariance of an ARMA process.
A linear process \( \{X_t\} \) is **causal** (strictly, a **causal function** of \( \{W_t\} \)) if there is a

\[
\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\psi_j| < \infty
\]

and

\[X_t = \psi(B)W_t.\]
A linear process \( \{X_t\} \) is invertible (strictly, an invertible function of \( \{W_t\} \)) if there is a

\[
\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\pi_j| < \infty
\]

and

\[W_t = \pi(B) X_t.\]
Review: AR(p), Autoregressive models of order $p$

An AR(p) process $\{X_t\}$ is a stationary process that satisfies

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t,$$

where $\{W_t\} \sim WN(0, \sigma^2)$.

Equivalently, $\phi(B) X_t = W_t$,

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$. 
Review: AR(p), Autoregressive models of order $p$

**Theorem:** A (unique) stationary solution to $\phi(B)X_t = W_t$ exists iff

$$|z| = 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$ 

This AR(p) process is causal iff

$$|z| \leq 1 \Rightarrow \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.$$
Polynomials of a complex variable

Every degree $p$ polynomial $a(z)$ can be factorized as

$$a(z) = a_0 + a_1 z + \cdots + a_p z^p = a_p (z - z_1)(z - z_2) \cdots (z - z_p),$$

where $z_1, \ldots, z_p \in \mathbb{C}$ are called the roots of $a(z)$. If the coefficients $a_0, a_1, \ldots, a_p$ are all real, then $c$ is real, and the roots are all either real or come in complex conjugate pairs, $z_i = \bar{z}_j$.

**Example:** $z + z^3 = z(1 + z^2) = (z - 0)(z - i)(z + i)$,

that is, $c = 1$, $z_1 = 0$, $z_2 = i$, $z_3 = -i$. So $z_1 \in \mathbb{R}$; $z_2, z_3 \notin \mathbb{R}$; $z_2 = \bar{z}_3$.

Recall notation: A complex number $z = a + ib$ has $\text{Re}(z) = a$, $\text{Im}(z) = b$, $\bar{z} = a - ib$, $|z| = a^2 + b^2$, $\text{arg}(z) = \tan^{-1}(b/a) \in (-\pi, \pi]$. 
Calculating $\psi$ for an AR(p): matching coefficients

Example: $X_t = \psi(B)W_t \iff (1 - 0.5B + 0.6B^2)X_t = W_t$, so $1 = \psi(B)(1 - 0.5B + 0.6B^2)$

$\iff 1 = (\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots)(1 - 0.5B + 0.6B^2)$

$\iff 1 = \psi_0,$

$0 = \psi_1 - 0.5\psi_0,$

$0 = \psi_2 - 0.5\psi_1 + 0.6\psi_0,$

$0 = \psi_3 - 0.5\psi_2 + 0.6\psi_1,$

$\vdots$
Calculating $\psi$ for an AR(p): example

\[
\Rightarrow \quad 1 = \psi_0, \quad 0 = \psi_j \quad (j \leq 0),
\]
\[
0 = \psi_j - 0.5\psi_{j-1} + 0.6\psi_{j-2}
\]
\[
\Leftrightarrow \quad 1 = \psi_0, \quad 0 = \psi_j \quad (j \leq 0),
\]
\[
0 = \phi(B)\psi_j.
\]

We can solve these linear difference equations in several ways:

- numerically, or
- by guessing the form of a solution and using an inductive proof, or
- by using the theory of linear difference equations.
Calculating $\psi$ for an AR(p): general case

$$\phi(B)X_t = W_t, \quad \Leftrightarrow \quad X_t = \psi(B)W_t$$

so

$$1 = \psi(B)\phi(B)$$

$$\Leftrightarrow \quad 1 = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p)$$

$$\Leftrightarrow \quad 1 = \psi_0,$$

$$0 = \psi_1 - \phi_1 \psi_0,$$

$$0 = \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0,$$

$$\vdots$$

$$\Leftrightarrow \quad 1 = \psi_0, \quad 0 = \psi_j \quad (j < 0),$$

$$0 = \phi(B)\psi_j.$$
**ARMA(p,q): Autoregressive moving average models**

An **ARMA(p,q)** process \( \{X_t\} \) is a stationary process that satisfies

\[ X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}, \]

where \( \{W_t\} \sim WN(0, \sigma^2) \).

- **AR(p)** = **ARMA(p,0)**: \( \theta(B) = 1 \).
- **MA(q)** = **ARMA(0,q)**: \( \phi(B) = 1 \).
ARMA processes

Can accurately approximate many stationary processes:

For any stationary process with autocovariance $\gamma$, and any $k > 0$, there is an ARMA process $\{X_t\}$ for which

$$\gamma_X(h) = \gamma(h), \quad h = 0, 1, \ldots, k.$$
ARMA(p,q): Autoregressive moving average models

An ARMA(p,q) process \( \{X_t\} \) is a stationary process that satisfies

\[
X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q},
\]

where \( \{W_t\} \sim WN(0, \sigma^2) \).

Usually, we insist that \( \phi_p, \theta_q \neq 0 \) and that the polynomials

\[
\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \quad \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q
\]

have no common factors. This implies it is not a lower order ARMA model.
ARMA(p,q): An example of parameter redundancy

Consider a white noise process $W_t$. We can write

$$X_t = W_t$$

$$\Rightarrow X_t - X_{t-1} + 0.25X_{t-2} = W_t - W_{t-1} + 0.25W_{t-2}$$

$$(1 - B + 0.25B^2)X_t = (1 - B + 0.25B^2)W_t$$

This is in the form of an ARMA(2,2) process, with

$$\phi(B) = 1 - B + 0.25B^2, \quad \theta(B) = 1 - B + 0.25B^2.$$  

But it is white noise.
**ARMA(p,q): An example of parameter redundancy**

ARMA model: \[ \phi(B)X_t = \theta(B)W_t, \]

with \[ \phi(B) = 1 - B + 0.25B^2, \]
\[ \theta(B) = 1 - B + 0.25B^2 \]

\[ X_t = \psi(B)W_t \]

\[ \Leftrightarrow \psi(B) = \frac{\theta(B)}{\phi(B)} = 1. \]

i.e., \( X_t = W_t. \)
Theorem: If \( \phi \) and \( \theta \) have no common factors, a (unique) stationary solution to \( \phi(B)X_t = \theta(B)W_t \) exists iff

\[
|z| = 1 \implies \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.
\]

This ARMA(p,q) process is causal iff

\[
|z| \leq 1 \implies \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0.
\]

It is invertible iff

\[
|z| \leq 1 \implies \theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0.
\]
Recall: Causality and Invertibility

A linear process \( \{X_t\} \) is **causal** if there is a

\[
\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\psi_j| < \infty \quad \text{and} \quad X_t = \psi(B)W_t.
\]

It is **invertible** if there is a

\[
\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots
\]

with

\[
\sum_{j=0}^{\infty} |\pi_j| < \infty \quad \text{and} \quad W_t = \pi(B)X_t.
\]
Example: \[(1 - 1.5B)X_t = (1 + 0.2B)W_t.\]

\[
\phi(z) = 1 - 1.5z = -\frac{3}{2} \left( z - \frac{2}{3} \right),
\]

\[
\theta(z) = 1 + 0.2z = \frac{1}{5} (z + 5).
\]

1. \(\phi\) and \(\theta\) have no common factors, and \(\phi\)’s root is at \(2/3\), which is not on the unit circle, so \(\{X_t\}\) is an ARMA(1,1) process.
2. \(\phi\)’s root (at \(2/3\)) is inside the unit circle, so \(\{X_t\}\) is not causal.
3. \(\theta\)’s root is at \(-5\), which is outside the unit circle, so \(\{X_t\}\) is invertible.
Example: $(1 + 0.25B^2)X_t = (1 + 2B)W_t.$

$$\phi(z) = 1 + 0.25z^2 = \frac{1}{4} (z^2 + 4) = \frac{1}{4}(z + 2i)(z - 2i),$$

$$\theta(z) = 1 + 2z = 2 \left(z + \frac{1}{2}\right).$$

1. $\phi$ and $\theta$ have no common factors, and $\phi$’s roots are at $\pm 2i$, which is not on the unit circle, so $\{X_t\}$ is an ARMA(2,1) process.
2. $\phi$’s roots (at $\pm 2i$) are outside the unit circle, so $\{X_t\}$ is causal.
3. $\theta$’s root (at $-1/2$) is inside the unit circle, so $\{X_t\}$ is not invertible.
Causality and Invertibility

**Theorem:** Let \( \{X_t\} \) be an ARMA process defined by \( \phi(B)X_t = \theta(B)W_t \). If \( \theta(z) \neq 0 \) for all \( |z| = 1 \), then there are polynomials \( \tilde{\phi} \) and \( \tilde{\theta} \) and a white noise sequence \( \tilde{W}_t \) such that \( \{X_t\} \) satisfies \( \tilde{\phi}(B)X_t = \tilde{\theta}(B)\tilde{W}_t \), and this is a causal, invertible ARMA process.

So we’ll stick to causal, invertible ARMA processes.
Calculating $\psi$ for an ARMA(p,q): matching coefficients

Example: $X_t = \psi(B)W_t \iff (1 + 0.25B^2)X_t = (1 + 0.2B)W_t$

so $1 + 0.2B = (1 + 0.25B^2)\psi(B)$

$\iff 1 + 0.2B = (1 + 0.25B^2)(\psi_0 + \psi_1B + \psi_2B^2 + \cdots)$

$\iff 1 = \psi_0,$

$0.2 = \psi_1,$

$0 = \psi_2 + 0.25\psi_0,$

$0 = \psi_3 + 0.25\psi_1,$

$\vdots$
Calculating $\psi$ for an ARMA(p,q): example

\[ 1 = \psi_0, \quad 0.2 = \psi_1, \]
\[ 0 = \psi_j + 0.25\psi_{j-2}. \]

We can think of this as $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$.

This is a first order difference equation in the $\psi_j$s.

We can use the $\theta_j$s to give the initial conditions and solve it using the theory of homogeneous difference equations.

$\psi_j = (1, \frac{1}{5}, -\frac{1}{4}, -\frac{1}{20}, \frac{1}{16}, \frac{1}{80}, -\frac{1}{64}, -\frac{1}{320}, \ldots)$. 
Calculating $\psi$ for an ARMA(p,q): general case

\[ \phi(B)X_t = \theta(B)W_t, \quad \Leftrightarrow \quad X_t = \psi(B)W_t \]

so \[ \theta(B) = \psi(B)\phi(B) \]

\[ \Leftrightarrow \quad 1 + \theta_1 B + \cdots + \theta_q B^q = (\psi_0 + \psi_1 B + \cdots)(1 - \phi_1 B - \cdots - \phi_p B^p) \]

\[ \Leftrightarrow \quad 1 = \psi_0, \]

\[ \theta_1 = \psi_1 - \phi_1 \psi_0, \]

\[ \theta_2 = \psi_2 - \phi_1 \psi_1 - \cdots - \phi_2 \psi_0, \]

\[ \vdots \]

This is equivalent to $\theta_j = \phi(B)\psi_j$, with $\theta_0 = 1$, $\theta_j = 0$ for $j < 0$, $j > q$. 