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Introduction to Time Series Analysis. Lecture 4.

1. Review: ACF, sample ACF.
2. Properties of the sample ACF.
3. Convergence in mean square.
A time series has mean function \( \{ X_t \} \) is stationary if both are independent of \( t \).

\[
\text{corr}(\tau X, \eta X) = \frac{\langle X_0 \rangle X_{\tau \eta}}{\langle X_\eta \rangle X_{\tau \eta}} = \langle \eta \rangle X_{\tau \eta}
\]

The autocorrelation function (ACF) is

\[
\text{cov}(\tau X, \eta X) = \langle \tau \eta + \tau + \eta X \rangle X_{\tau \eta}
\]

and autocovariance function

\[
[\tau X] \text{E} = \tau \eta
\]

Mean, Autocovariance, Stationarity
For observations $x_1, \ldots, x_n$ of a time series, the sample mean is $\bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t$.

The sample autocovariance function is

$$\gamma(h) = \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})(x_{t+h} - \bar{x})$$

for $n \geq h < 2n$.

The sample autocorrelation function is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

for $h \neq 0$. 

For observations $x_1, \ldots, x_n$ of a time series, the sample mean is $\bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t$.
Properties of the autocovariance function

For the autocovariance function of a stationary time series, $\{X_t\}$, it follows that there exists a function $\gamma$ in $\mathbb{R} \rightarrow \mathbb{R}$ that satisfies (3) and (4) and (4) is the autocovariance of some stationary (Gaussian) time series.

Furthermore, any function $\gamma$ in $\mathbb{R} \rightarrow \mathbb{R}$ that satisfies (3) and (4) is positive semidefinite.

1. $\gamma$ is positive semidefinite.
2. $(\gamma - (\gamma))\gamma \geq 0$.
3. $(\gamma)\gamma = (\gamma)\gamma$.
4. $(0)\gamma \geq \gamma(0)$.

For the autocovariance function of a stationary time series, $\gamma$ of $\{X_t\}$, it follows that there exists a function $\gamma$ in $\mathbb{R} \rightarrow \mathbb{R}$ that satisfies (3) and (4) and (4) is the autocovariance of some stationary (Gaussian) time series.
The sample autocovariance function:

\[ (h) = \frac{1}{n} \sum_{j=1}^{n-j} (x_t + jh)(x_t) \] 

for \( n < h < n \).

For any sequence \( x_1, \ldots, x_n \), the sample autocovariance function satisfies:

1. \( (0) \geq |(h)| \) and \( 0 \leq (0) \)
2. \( (\hat{y} - h) = (\hat{y}) \)
3. \( (0) \geq |(h)| \) and hence \( (\hat{y} - h) = (\hat{y}) \).

For any sequence \( x_1, \ldots, x_n \), the sample autocovariance function satisfies:

\[ u > \eta > u- \quad \text{for} \quad (x - \hat{x})(x - \text{floor}(\eta + h) \sum_{\eta = u}^{1} \frac{u}{1} = (\eta) \]
Properties of the sample autocovariance function: psd

\[ 0 \lesssim \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (0)^n \\ (1)^n \end{bmatrix} \]

\[
(p, I) (I, p) \begin{bmatrix} u \\ v \end{bmatrix} = p^{u \downarrow, p} \]

\[
\begin{bmatrix} (0)^n \\ (1)^n \end{bmatrix} = \begin{bmatrix} (0)^n \\ (1)^n \end{bmatrix} \begin{bmatrix} (0)^n \\ (1)^n \end{bmatrix} = u^{u \downarrow} \]

\[
\begin{pmatrix} (0)^n & \cdots & (z - u)^n & (1 - u)^n \\ \vdots & \vdots & \vdots & \vdots \\ (z - u)^n & \cdots & (0)^n & (1)^n \\ (1 - u)^n & \cdots & (1)^n & (0)^n \end{pmatrix} = u^{u \downarrow} \]
Properties of the sample autocovariance function: psd

\[ r_l - t X = t X \]

\[
\begin{pmatrix}
0 & \cdots & 0 & 0 & u_X & \cdots & z_X & i_X \\
\vdots & & & & \vdots & & & \\
0 & 0 & u_X & \cdots & z_X & i_X & \cdots & 0 \\
0 & u_X & \cdots & z_X & i_X & 0 & \cdots & 0 \\
u_X & \cdots & z_X & i_X & 0 & 0 & \cdots & 0
\end{pmatrix} = W
\]
For a stationary process, the sample average \( \{ X_t \} \) satisfies

\[
\mathbb{E} \left( u X + \cdots + X \right) \frac{u}{1} = u X
\]
Estimating the ACF: Sample ACF

\[ \hat{\rho}(k) = \frac{1}{n-k} \sum_{t=k+1}^{n} \frac{X_t - \bar{X}}{\sigma} \left( \frac{X_{t-k} - \bar{X}}{\sigma} \right) \]

\[ \text{var}(X) = \mathbb{E}(X - \bar{X})^2 = \text{var}(X) \]
Theorem 1.5

For a linear process $X_t = P_j W_t$, if $P_j \neq 0$, then

$E \sum_{j=-\infty}^{\infty} x_t \phi_j = \sum_{j=-\infty}^{\infty} \phi_j x_t + \eta = X_t$
Recall: for a linear process

Estimating the ACF: Sample ACF
Theorem 1.7

For a linear process \( X_t = \sum_j W_j \), if \( E(W_t^4) < 1 \),

\[
\begin{gathered}
I = \Lambda, 0 \neq \gamma \quad \text{for all } 0 = (\gamma)^d \\
\cdot ((\gamma)d(\gamma)dZ - (\gamma - \gamma)d + (\gamma + \gamma)d) \times \\
\sum_{i=\gamma}^{\infty} \sum_{j=\gamma}^{\infty} = \gamma \Lambda
\end{gathered}
\]

where

\[
\begin{pmatrix}
\Lambda^{-1} & \left( \begin{array}{c}
(Y)^d \\
\vdots \\
(I)^d
\end{array} \right) \\
\end{pmatrix}
\]

\( NV \sim \left( \begin{array}{c}
(Y)^d \\
\vdots \\
(I)^d
\end{array} \right)
\]

\[
\begin{pmatrix}
\Lambda^{-1} \\
\end{pmatrix}
\]

\( E(M_{i}W_{j}^d) \sum + \eta = X \)

**Estimating the ACF: Sample ACF**
If $f_X$ is white noise, we expect no more than \(5\%\) of the peaks of the sample ACF to satisfy:

\[
\frac{u^\wedge}{96} < |(y)\xi|
\]

reduce a time series to white noise.

This is useful because we often want to introduce transformations that

If \(\{X\}\) is white noise, we expect no more than \(5\%\) of the peaks of the sample ACF and testing for white noise.
Sample ACF for white Gaussian (hence i.i.d.) noise
Sample ACF for MA(1)

Recall:

\[(\theta) = 1, \ \theta = 2,\] and \((\theta + 1)^{\theta} = (\theta + 1)\) for \(\theta \)

Thus, \(\theta < 1\) for \(\theta = 0\).
Sample ACF for MA(1)
Convergence in Mean Square

A sequence of random variables \( S_1, S_2, \ldots \) converges in mean square if there is a random variable \( Y \) for which

\[
\lim_{n \to \infty} \mathbb{E} \left( (S_n - Y)^2 \right) = 0.
\]

The Riesz-Fisher Theorem (Cauchy criterion):

\[
0 = \lim_{m,n \to \infty} \mathbb{E} (S_m - S_n)^2.
\]
Example: Linear Process

\[
X_t = (B) \phi_t
\]

where

\[
\sum_{t=0}^{\infty} (B) \phi_t = \phi_t X
\]
Example: Linear Process

\[ L(\mathbf{x}_t) = \mathbf{x}_t \]

1. \[ \mathbb{E} |\mathbb{E} L(\mathbf{x}_t) | \leq \rho \]

2. \[ \mathbb{E} \rho \leq \mathbb{E} |\mathbb{E} L(\mathbf{x}_t) | \]

Jensen's Inequality

Markov's Inequality
Example: Linear Process

\[ S_n = \sum_{j=1}^{n} X_j \]

converges in mean square, since

\[ E(S_m - S_n)^2 = \sum_{j=1}^{m} \sum_{k=1}^{n} E(X_j X_k) \]

where \( E(X_j X_k) = \begin{cases} 0 & \text{if } j \neq k, \\ E(X_j^2) & \text{if } j = k. \end{cases} \]
$\infty > |\phi| \quad \sum_{t=0}^{\infty} > |\phi| > |\phi|$ implies, since it converges in mean square, that is a solution. The same argument as before shows that this infinite sum is a solution.

\[
\begin{align*}
\sum_{t=0}^{\infty} M_t \phi &= X \\
\sum_{t=0}^{\infty} M_t \phi &= X
\end{align*}
\]

\[
\sum_{t=0}^{\infty} M_t \phi = X
\]

Let $\mathbb{X}$ be the stationary solution to $X$: $\mathbb{X}$. Where $X = \sum_{t=0}^{\infty} X \phi - \mathbb{X}$.
Furthermore, $X_t$ is the unique stationary solution: we can check that any other stationary solution $Y_t$ is the mean square limit:

$$0 = \lim_{n \to \infty} \mathbb{E} \left[ (Y_t - X_t) \right]$$
Example: AR(1)

\[ rM(B) = rX \quad \Leftrightarrow \]
\[ rM(B) = rX(B)\phi(B) \]

Thus,
\[ rM = rX(B)\phi \]

\[ rI = r\sum_{i=0}^{\infty} \phi B_i \phi \]
\[ r\sum_{i=0}^{\infty} \phi B_i \phi = (\phi - 1) r\sum_{i=0}^{\infty} \phi B_i \phi = (B)\phi(B) \]

Then we can check that
\[ (B)\phi = (B) \]

Equivalently, if we write
\[ rI = \phi B \]
\[ r\sum_{i=0}^{\infty} \phi B_i = (B) \]
Example: AR(1)

Notice that manipulating operators like $\phi(B)$, $\pi(B)$ is like manipulating polynomials:

\[
\frac{1}{1 - \phi z} = 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \ldots ,
\]

provided $|\phi| < 1$ and $|z| < 1$. 

Example: AR(1)

Let $X_t$ be the stationary solution to

$$X_t - \phi X_{t-1} = W_t;$$

where $W_t \sim WN(0, \sigma^2)$.

If $|\phi| < 1$,

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}. $$

If $|\phi| = 1$?

If $|\phi| = 1$?

If $|\phi| > 1$?