Stat 134 (P2): Concepts of Probability, UC Berkeley, Spring 2013

Problem Set 5

Instructor: Prof. Yun S. Song Due: February 28, 2013, in the beginning of class.

1. A real-valued function f over $[a, b] \subset \mathbb{R}$ is said to be *convex* if

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2),$$

for all $x_1, x_2 \in [a, b]$ and $0 < \lambda < 1$. Let X denote a discrete random variable that takes values in $\{0, 1, \ldots, n\}$. Show that, for f a convex function over $[0, n]$,

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}(X)).$$

2. Consider a biased coin that shows heads (H) with probability p and tails (T) with probability $1-p$. Suppose the coin is tossed n times. A run of k heads refers to consecutive occurrences of *exactly* k heads. That is, a run of 2 heads at the beginning results in the pattern "$HHT\ldots$", at the end results in the pattern "$\ldots THH$", and elsewhere in the sequence leads to "$\ldots THHT\ldots$". For $1 \leq k \leq n$, let $X_{n,k}$ denote the number of runs of k heads in n tosses.

   (a) For $1 \leq k \leq n$, find $\mathbb{E}(X_{n,k})$.

   (b) Let Y_n denote the total number of non-overlapping runs of heads in n tosses, where runs are of any length between 1 and n. Using (a), find $\mathbb{E}(Y_n)$.

   (c) For $1 \leq i \leq n$, what is the probability that a run of heads of some length starts on the ith toss?

   (d) Using (c), find $\mathbb{E}(Y_n)$. Check that your answers to (b) and (d) agree.

3. Do the following problems from the textbook: 3.3 (8, 14, 16); 3.4 (4, 10)