Stat 134 (P2): Concepts of Probability, UC Berkeley, Spring 2013

Problem Set 10

Instructor: Prof. Yun S. Song — Due: April 18, 2013

Show all your work to receive full credit.

1. [6 Points] Consider a random arrival process on $(0, \infty)$. For fixed a, b satisfying $0 < a < b$, let $N(a, b]$ denote the number of arrivals in the time interval $(a, b]$. Furthermore, let W_1 denote the waiting time to the first arrival, and, for $i > 1$, let W_i denote the waiting time between the $(i-1)$th and the ith arrivals.

   (a) [3 Points] For all $k \geq 1$, suppose W_k are i.i.d. random variables distributed as $\text{Exp}(\lambda)$, where $\lambda > 0$. Show that this implies $N(a, b] \sim \text{Poisson}(\lambda(b-a))$, where $0 < a < b$.

   (b) [3 Points] Assuming the condition stated in part (a), find the conditional distribution of $N(0, s]$ given $N(0, t] = n$, where $0 < s < t$ and $0 \leq m \leq n$. What is the name of this distribution?

2. [9 Points] Suppose X and Y are i.i.d. normal random variables with mean μ and variance σ^2. Find the following probabilities in terms of the c.d.f. Φ of the standard normal distribution:

   (a) [3 Points] $\mathbb{P}(2X < Y + \mu)$.

   (b) [3 Points] $\mathbb{P}(|\min(X, Y) - \mu| < \sigma)$.

   (c) [3 Points] $\mathbb{P}(\max(X, Y) - \min(X, Y) < 2\sigma)$.

3. [9 Points] Let $(X_1, Y_1), (X_2, Y_2), (X_3, Y_3)$ denote three random points in $\mathbb{R}^2$, where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are independent standard normal random variables. Let D denote the distance between (X_1, Y_1) and (X_2, Y_2).

   (a) [3 Points] Find the density and c.d.f. of D. What is the name of this distribution?

   (b) [3 Points] Find $\mathbb{E}(D)$ and $\text{Var}(D)$.

   (c) [3 Points] Let C be the circle that contains (X_1, Y_1) and (X_2, Y_2) as antipodal (i.e., diametrically opposite) points. What is the probability that (X_3, Y_3) lies inside C?

4. [3 Points] For $\alpha, \beta > 0$, let $X \sim \text{Exp}(\alpha)$ and $Y \sim \text{Exp}(\beta)$ be independent random variables. Find the c.d.f. of $Z = X/Y$.

5. [6 Points] For $\alpha, \beta > 0$, let $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ be independent random variables.

   (a) [3 Points] Show that $(X + Y)$ and $X/(X + Y)$ are independent.

   (b) [3 Points] For $\alpha, \beta > 0$, recall that the $\text{Beta}(\alpha, \beta)$ distribution on the interval $(0, 1)$ is defined by the density

   $$f(z) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} z^{\alpha-1}(1-z)^{\beta-1},$$

   for $0 < z < 1$. (See Pitman pages 327–329 for further details.) Show that $X/(X + Y)$ has $\text{Beta}(\alpha, \beta)$ distribution.