

Stat 155 Fall 08 Solution to HW 7

1. The following computations are straightforward.

$$\begin{aligned}
 c_{\{1\}} &= v(\{1\}) = 1, & c_{\{2\}} &= v(\{2\}) = 0, & c_{\{3\}} &= v(\{3\}) = -4. \\
 c_{\{1,2\}} &= v(\{1,2\}) - c_{\{1\}} - c_{\{2\}} = 1, \\
 c_{\{1,3\}} &= v(\{1,3\}) - c_{\{1\}} - c_{\{3\}} = 2, \\
 c_{\{2,3\}} &= v(\{2,3\}) - c_{\{2\}} - c_{\{3\}} = 7, \\
 c_{\{1,2,3\}} &= v(\{1,2,3\}) - c_{\{1,2\}} - c_{\{1,3\}} - c_{\{2,3\}} - c_{\{1\}} - c_{\{2\}} - c_{\{3\}} \\
 &= -1.
 \end{aligned}$$

Thus, we have

$$\begin{aligned}
 \phi_1(v) &= c_{\{1\}} + \frac{c_{\{1,2\}}}{2} + \frac{c_{\{1,3\}}}{2} + \frac{c_{\{1,2,3\}}}{3} = \frac{13}{6}. \\
 \phi_2(v) &= c_{\{2\}} + \frac{c_{\{1,2\}}}{2} + \frac{c_{\{2,3\}}}{2} + \frac{c_{\{1,2,3\}}}{3} = \frac{11}{3}. \\
 \phi_3(v) &= c_{\{3\}} + \frac{c_{\{1,3\}}}{2} + \frac{c_{\{2,3\}}}{2} + \frac{c_{\{1,2,3\}}}{3} = \frac{1}{6}.
 \end{aligned}$$

2. We know that (Ferguson, Theorem 2, page IV - 15)

$$\phi_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!} (v(S) - v(S \setminus \{i\})).$$

For any set S containing i , the superadditivity property of the characteristic function v implies that $v(\{i\}) + v(S \setminus \{i\}) \leq v(S)$. From this and the above formula, we see that

$$\phi_i(v) \geq \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!} v(\{i\}). \tag{1}$$

It is a simple combinatorial exercise to verify that

$$\sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!} = 1. \quad (2)$$

This is also obvious from the fact that if the players are arranged in a random order, and S is the set containing i and all players that come before i in the random arrangement, then for any set S containing i

$$\mathbb{P}(S = S) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|S| - 1)!(n - |S|)!}{n!},$$

which proves (2) because the probabilities of mutually exclusive and exhaustive events must sum to 1. Combining (1) and (2), we get $\phi_i(v) \geq v(\{i\})$, which proves that the Shapley value is individually rational. Group rationality is the same as the efficiency axiom of the Shapley value.

3. Clearly, we have

$$\begin{aligned} v(\{1\}) &= v(\{2\}) = v(\{3\}) = v(\{4\}) = 0. \\ v(\{1, 2\}) &= v(\{1, 3\}) = v(\{1, 4\}) = 0. \\ v(\{2, 3\}) &= v(\{2, 4\}) = v(\{3, 4\}) = 1. \\ v(\{1, 2, 3\}) &= v(\{1, 2, 4\}) = v(\{1, 3, 4\}) = v(\{2, 3, 4\}) = 1. \\ v(\{1, 2, 3, 4\}) &= 1. \end{aligned}$$

A set S is pivotal for player i if S is a winning set containing i and $S \setminus \{i\}$ is a losing set. The pivotal sets for the four players are as follows.

Player 1 : None.

Player 2 : $\{2, 3\}$, $\{2, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$.

Player 3 : $\{2, 3\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$.

Player 4 : $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$.

Thus from formula (7) in Ferguson, page IV - 17, we get

$$\begin{aligned} \phi_1(v) &= 0. \\ \phi_2(v) &= 2 \frac{(2-1)!(4-2)!}{4!} + 2 \frac{(3-1)!(4-3)!}{4!} = \frac{1}{3}. \\ \phi_3(v) &= 2 \frac{(2-1)!(4-2)!}{4!} + 2 \frac{(3-1)!(4-3)!}{4!} = \frac{1}{3}. \\ \phi_4(v) &= 2 \frac{(2-1)!(4-2)!}{4!} + 2 \frac{(3-1)!(4-3)!}{4!} = \frac{1}{3}. \end{aligned}$$

4. If the players are numbered as $1, 2, \dots, 15$, with $1, \dots, 5$ representing the big five. By symmetry, it suffices to compute $\phi_1(v)$ and $\phi_6(v)$. Moreover, by the efficiency axiom,

$$5\phi_1(v) + 10\phi_6(v) = 1, \quad (3)$$

and so it suffices to simply compute $\phi_1(v)$. The pivotal sets for player 1 are precisely the sets S that contain all the big five and at least four of the rest. There are $\binom{10}{k}$ sets containing all the big five and k of the rest. Thus, from formula (7) in Ferguson, page IV - 17,

$$\begin{aligned} \phi_1(v) &= \sum_{k=4}^{10} \binom{10}{k} \frac{(5+k-1)!(15-(5+k))!}{15!} \\ &= \sum_{k=4}^{10} \frac{10!}{k!} \frac{(k+4)!}{15!} \\ &= \sum_{k=4}^{10} \frac{(k+1)(k+2)(k+3)(k+4)}{11 \cdot 12 \cdot 13 \cdot 14 \cdot 15} \approx 0.196. \end{aligned}$$

Using this and the equation (3) above, we get that for $i = 1, \dots, 5$, $\phi_i(v) \approx 0.196$ and for $i = 6, \dots, 15$, $\phi_i(v) \approx 0.002$. Thus, the smaller nations have almost no power at all.

5. We will proceed by computing c_S for $S \subseteq N$. Clearly, $c_{\{1\}} = 1$ and $c_{\{i\}} = 0$ for $i \geq 2$. We claim that $c_S = 1$ if S is of the form $\{1, 2, \dots, k\}$ for some $k \in \{1, \dots, n\}$, and $c_S = 0$ for all other S . Let us prove this by induction on the size of S . We have already observed this for sets of size 1. Suppose this claim has been proved for all sets of size $\leq k-1$. Let S be a set of size k . Let l be the smallest integer not in S . We want to show that $c_S = 0$ if $l \leq k$ and $c_S = 1$ if $l = k+1$. First, if $l = k+1$, then $S = \{1, \dots, k\}$, and so by the induction hypothesis, there are exactly $k-1$ proper subsets T of S such $c_T = 1$, and the rest are zero. Thus, for $S = \{1, \dots, k\}$,

$$c_S = v(S) - \sum_{T \subsetneq S} c_T = k - (k-1) = 1.$$

Next, consider the case $l \leq k$. By the induction hypothesis, we see that there are exactly $l-1$ proper subsets T of S such that $c_T = 1$. Thus,

$$c_S = v(S) - \sum_{T \subsetneq S} c_T = (l-1) - (l-1) = 0.$$

This completes the proof of the claim about the coefficients c_S . Using this, we get that for any i ,

$$\phi_i(v) = \sum_{k=i}^n \frac{c_{\{1, \dots, k\}}}{k} = \sum_{k=i}^n \frac{1}{k}.$$

6. Let $c_0 = 0$. Let $R_k = \{k, k+1, \dots, n\}$ and define the function v_k as

$$v_k(S) = \begin{cases} -(c_k - c_{k-1}) & \text{if } S \cap R_k \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $v_k(\emptyset) = 0$ and v_k is superadditive (here we use the assumption that $c_{k-1} < c_k$). Thus, v_k is a characteristic function. Now, for any S , if j is the maximum element of S , then $v_k(S) = -(c_k - c_{k-1})$ for any $k \leq j$, and $v_k(S) = 0$ for any $k > j$. Thus,

$$v(S) = \sum_{k=1}^n v_k(S).$$

Therefore, by the additivity of the Shapley value, we have that for any $i \in \{1, \dots, n\}$,

$$\phi_i(v) = \sum_{k=1}^n \phi_i(v_k).$$

Let us now compute $\phi_i(v_k)$. Fix k . If $i < k$, then for any S containing i , we have $v_k(S) = v_k(S \setminus \{i\})$. Thus, by the dummy axiom, $\phi_i(v_k) = 0$ whenever $i < k$. Next, note that for any $i, j \geq k$, and any set S not containing i and j , it is easy to see that $v_k(S \cup \{i\}) = v_k(S \cup \{j\})$. Thus, for all $i, j \geq k$, $\phi_i(v_k) = \phi_j(v_k)$ by the symmetry axiom. Finally, observe that by the efficiency axiom,

$$\sum_{i=1}^n \phi_i(v_k) = v_k(\{1, \dots, n\}) = -(c_k - c_{k-1}).$$

Combining, the three observations, we get that for any $i \geq k$,

$$\phi_i(v_k) = -\frac{c_k - c_{k-1}}{n - k + 1}.$$

Thus, for any i ,

$$\phi_i(v) = -\sum_{k=1}^i \frac{c_k - c_{k-1}}{n - k + 1}.$$