Stat 155 Fall 08 Practice Final

December 9, 2008

Time: 3 hours.
Please show all steps.

1. Consider a game of nim with inital configuration $(48, 23, 74, 10)$ and with the restriction that at most three chips can be removed from a pile in a single move. Is this an N position or a P position? If it is an N position, what is a winning first move?

2. Consider the following two-person impartial combinatorial game: starting with a single pile of n chips, the players take turns at removing chips from the pile, subject to the following rule. At each move, a player is allowed to remove k chips from the pile only if k is a divisor of the number of chips remaining in the pile. The player to remove the last chip wins. Compute the Sprague-Grundy function. (First try to guess it by doing some computations and then try to prove by induction.)

3. Assuming the separating hyperplane theorem, prove von Neumann's minimax theorem.

4. Compute the optimal strategies and the value of the two-person zero sum game with payoff matrix:

$$\begin{pmatrix} 4 & 1 & -1 \\ 0 & 3 & 4 \end{pmatrix}$$

5. There are two gates in a medieval fort, and the captain of the fort has resources to guard only one against an enemy attack. The enemy has n soldiers, and he can decide on how to split up his soldiers between the gates. The payoff to the captain is the number of enemy soldiers he can intercept. Thus, if he is guarding gate 1, and the enemy sends

30 soldiers to gate 1 and $n - 30$ to gate 2, then the payoff to player I (the captain) is 30. Considering this as a two-person zero sum game, guess the optimal strategies and prove that they are indeed optimal.

6. Find all Nash equilibria in the general sum game:

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix}$$

7. In the parliament of a certain country, there are four political parties with 80, 50, 30, and 10 seats respectively. A coalition that has more than half of the members of the parliament can pass a certain bill. Compute the Shapley-Shubik power index for the four parties. (Note: Whenever you compute the Shapley-Shubik index, please make sure to verify that the Shapley values add up to one, just to be certain that your computations are correct.)

8. Suppose that a certain social ranking rule satisfies Arrow's fairness criteria. Suppose an alternative b is ranked as the top or bottom candidate by each individual. Show that b must be at the top or bottom in the social ranking also.

9. Show that the Gale-Shapley algorithm for finding a stable matching between n men and n women must converge within $n^2 - n$ steps.