Stat 155 Fall 08 Practice Final Solutions

December 15, 2008

1. If g_i is the Sprague-Grundy function for the game consisting of only the ith pile, a simple computation shows that for any n, $g_i(n) =$ the remainder when n is divided by 4. Thus, if g is the Sprague-Grundy function of the game with four piles, then by the Sprague-Grundy theorem,

$$\begin{aligned} g(48, 23, 74, 10) &= g_1(48) \oplus g_2(23) \oplus g_3(74) \oplus g_4(10) \\ &= 0 \oplus 3 \oplus 2 \oplus 2 = 3 \neq 0. \end{aligned}$$

   Thus, this is an N position. Since $2 \oplus 2 = 0$, therefore if we remove three chips from pile 2 to get $(48, 20, 74, 10)$ as our new configuration, then the nim-sum becomes zero, and hence this is a winning move.

2. Define a function h as follows: Let $h(0) = 0$, and for any positive integer n, let $h(n) =$ the smallest positive integer k such that n is *not* divisible by 2^k. For example, $h(2) = 2$, $h(3) = 1$, $h(4) = 3$, etc. We claim that h is the Sprague-Grundy function of this game.

   Let g be the actual Sprague-Grundy function. Then $g(0) = h(0) = 0$ by definition. We have to show that $g(n) = h(n)$ for each n. We proceed by induction. First, note that $g(1) = 1$ by the definition of the SG function, and $h(1) = 1$ by the definition of h. Now take any n, and suppose $g(m) = h(m)$ for all $m < n$. Let us now show that $g(n) = h(n)$. We will use the definition

$$g(n) = \text{mex}\{g(m) : n \to m\},$$

   where $n \to m$ means that reducing the pile to size m from size n is a legal move (i.e. $n - m$ is a divisor of n).

   Let r be the largest nonnegative integer such that n is divisible by 2^r. Then $h(n) = r + 1$. Now n can be written as $n = 2^r m$, where m is an odd number. For any $0 \le s < r$, $2^s m$ is a divisor of n. Thus, the pile can be reduced to $2^r m - 2^s m$ in a single move. Now $2^r m - 2^s m = (2^{r-s}-1)2^s m$, and $2^{r-s}-1$ is odd. Hence, the largest power of 2 that divides $2^r m - 2^s m$ is s. By the induction hypothesis, this shows that $1, \ldots, r$ all belong to the set $\{g(m) : n \to m\}$. Since we are allowed to reduce the pile to zero and $g(0) = 0$, therefore 0 belongs to this set also. Now we only have to show that $r+1 \notin \{g(m) : n \to m\}$. The divisors of n are all of the form $2^s d$, where $0 \le s \le r$ and d is a proper divisor of m. Take any such divisor, and let $q = m/d$. Now,

$$2^r m - 2^s d = (2^{r-s} q - 1)2^s d.$$

If $s < r$, then $2^{r-s}q - 1$ is odd, and it follows that $g(2^r m - 2^s d) = s + 1 \le r$. On the other hand, if $s = r$, then $2^{r-s}q - 1 = q - 1$ is a positive even number, since q is odd and strictly bigger than 1. Thus, $g(2^r m - 2^r d) > r + 1$. This completes the argument.

3. See proof in text.

4. For any strategy $\boldsymbol{p} = (p, 1-p)$ of player I, the minimum assured expected gain is

$$\begin{aligned} f_{\mathrm{I}}(\boldsymbol{p}) &= \min\{4p,\ p + 3(1-p),\ -p + 4(1-p)\} \\ &= \min\{4p,\ 3 - 2p,\ 4 - 5p\}. \end{aligned}$$

A simple algebraic verification shows that the above minimum equals $4p$ is $0 \le p \le 4/9$, and equals $4 - 5p$ when $4/9 \le p \le 1$. Since $4p$ is an increasing function of p and $4 - 5p$ is a decreasing function of p, this shows that the maximum of $f_{\mathrm{I}}(\boldsymbol{p})$ is achieved when $p = 4/9$, that is, at $\boldsymbol{p} = (4/9, 5/9)$. Thus, the value of the game is $f_{\mathrm{I}}((4/9, 5/9)) = 16/9$.

For a strategy $\boldsymbol{q} = (q_1, q_2, 1 - q_1 - q_2)$, the maximum possible expected loss for II is

$$\begin{aligned} f_{\mathrm{II}}(\boldsymbol{q}) &= \max\{4q_1 + q_2 - (1 - q_1 - q_2),\ 3q_2 + 4(1 - q_1 - q_2)\} \\ &= \max\{5q_1 + 2q_2 - 1,\ 4 - 4q_1 - q_2\}. \end{aligned}$$

Now, if we can find $\boldsymbol{q}$ such that $f_{\mathrm{II}}(\boldsymbol{q}) = 16/9$, then by a lemma proved in class, $\boldsymbol{q}$ must be an optimal strategy. We find this $\boldsymbol{q}$ as follows. First, we narrow down our search space by equalizing the two terms in the above maximum, getting the relation

$$q_2 = \frac{5}{3} - 3q_1.$$

With this relation, the equation $f_{\mathrm{II}}(\boldsymbol{q}) = 16/9$ boils down to $q_1 = 5/9$, $q_2 = 0$. Finally, we verify that $f_{\mathrm{II}}((5/9, 0, 4/9)) = 16/9$, and hence $(5/9, 0, 4/9)$ is an optimal strategy for player II.

5. The payoff matrix is

$$\begin{pmatrix} n & n-1 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & n-1 & n \end{pmatrix}$$

For any strategy $\boldsymbol{p} = (p, 1-p)$ of player I, the minimum assured expected gain is

$$f_{\mathrm{I}}(\boldsymbol{p}) = \min\{np,\ (n-1)p + (1-p),\ \ldots,\ p + (n-1)(1-p),\ n(1-p)\}.$$

Thus,

$$f_{\mathrm{I}}((1/2, 1/2)) = \frac{n}{2}.$$

Again, for any strategy $\boldsymbol{q} = (q_n, q_{n-1}, \ldots, q_0)$ of II,

$$f_{\mathrm{II}}(\boldsymbol{q}) = \max\{nq_n + (n-1)q_{n-1} + \cdots + q_1,\ q_{n-1} + 2q_{n-2} + \cdots + nq_0\}.$$

Thus,

$$f_{\mathrm{II}}((\tfrac{1}{n+1}, \tfrac{1}{n+1}, \ldots, \tfrac{1}{n+1})) = \frac{n}{2}.$$

Since

$$f_{\mathrm{I}}((1/2, 1/2)) = f_{\mathrm{II}}((\tfrac{1}{n+1}, \tfrac{1}{n+1}, \ldots, \tfrac{1}{n+1})),$$

it follows that $(1/2, 1/2)$ is an optimal strategy for I and $(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1})$ is an optimal strategy for II.

6. For any $p, q \in [0, 1]$, let (p, q) denote the pair of strategies where Player I plays move 1 (i.e. the top row) with probability p and Player II plays his move 1 (i.e. the left column) with probability q. For any such a pair (p, q), the expected payoffs for I and II are

$$\begin{aligned} f_{\mathrm{I}}(p, q) &= 3 \cdot pq + 0 \cdot p(1-q) + 2 \cdot (1-p)q + 5 \cdot (1-p)(1-q) \\ &= p(6q - 5) + 5 - 3q, \\ f_{\mathrm{II}}(p, q) &= 3 \cdot pq + 2 \cdot p(1-q) + 1 \cdot (1-p)q + 5 \cdot (1-p)(1-q) \\ &= q(5p - 4) + 5 - 3p. \end{aligned}$$

Suppose (p, q) is a Nash equilibrium. Then for Player I, p is the optimal strategy against q. This means

(a) If $6q - 5 < 0$, then $p = 0$.

(b) If $6q - 5 > 0$, then $p = 1$.

(c) If $6q - 5 = 0$, then p can be anything.

Again, for Player II, q is the optimal strategy against p. Thus,

(a') If $5p - 4 < 0$, then $q = 0$.

(b') If $5p - 4 > 0$, then $q = 1$.

(c') If $5p - 4 = 0$, then q can be anything.

Now let us examine what are the possible Nash equilibria. Suppose $p = 0$. Then $5p - 4 < 0$, which implies that $q = 0$. With $q = 0$, we get $6q - 5 < 0$, which implies $p = 0$. Thus, there is no contradiction and $(0, 0)$ is indeed a Nash equilibrium. Similarly, $(1, 1)$ is a Nash equilibrium. Now suppose $0 < p < 1$. If $0 < p < 4/5$, then by (b'), $q = 0$. But $q = 0$ implies that $p = 0$, as we reasoned before. This shows that we cannot have $0 < p < 4/5$. Similarly, we can argue that $4/5 < p < 1$ is impossible, and the similar conclusions can be drawn for q as well. The final step is to check whether $(4/5, 5/6)$ is indeed a Nash equilibrium. This is true, by (c) and (c').

Combining all conclusions, we see that the game has three Nash equilibria, namely $(0, 0)$, $(1, 1)$, and $(4/5, 5/6)$.

7. Clearly, we have

$$
\begin{aligned}
&v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = 0. \\
&v(\{1,2\}) = 1,\ v(\{1,3\}) = 1,\ v(\{1,4\}) = 1. \\
&v(\{2,3\}) = v(\{2,4\}) = v(\{3,4\}) = 0. \\
&v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 1. \\
&v(\{1,2,3,4\}) = 1.
\end{aligned}
$$

A set S is pivotal for player i if S is a winning set containing i and $S \backslash \{i\}$ is a losing set. The pivotal sets for the four players are as follows.

$$
\begin{aligned}
&\text{Player 1}: \ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,2,3\},\ \{1,2,4\},\ \{1,3,4\}. \\
&\text{Player 2}: \ \{1,2\},\ \{2,3,4\}. \\
&\text{Player 3}: \ \{1,3\},\ \{2,3,4\}. \\
&\text{Player 4}: \ \{1,4\},\ \{2,3,4\}.
\end{aligned}
$$

Thus from formula (7) in Ferguson, page IV - 17, we get

$$\phi_1(v) = 3\frac{(2-1)!(4-2)!}{4!} + 3\frac{(3-1)!(4-3)!}{4!} = \frac{1}{2}.$$
$$\phi_2(v) = \phi_3(v) = \phi_4(v) = \frac{(2-1)!(4-2)!}{4!} + \frac{(3-1)!(4-3)!}{4!} = \frac{1}{6}.$$

8. See proof in text.

9. See proof in text.