HW #9

(1)

$$\begin{array}{cc} & \text{II} \\ \text{I} & \begin{pmatrix} (4,2) & (1,1) \\ (0,0) & (2,4) \end{pmatrix} \end{array}$$

$I = \{ \vec{x} \in \mathbb{R}^2 : \sum_{i=1}^{2} x_i = v(N),\ x_i \geq v_i\ \forall i \in N \}$

$C = \{ \vec{x} \in \mathbb{R}^2 : \sum x_i = v(N), \sum_{i \in S} x_i \geq v(S)\ \forall S \subseteq N \}$

$A = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}$

Note $v(N) = 6$ = highest sum of payoffs.

$v\phi = 0$

$v\{1\} = v\{2\} = 8/5$ (By computing the value of the game)

$\Rightarrow I = \{ (x,y) \in \mathbb{R}^2 : x+y = 6,\ x \geq 8/5,\ y \geq 8/5 \}$

In a two person game, $I = C$, since only $S \subseteq N$ are $\phi, \{1\}, \{2\}, N$

$\Rightarrow C = \{ (x, 6-x) \in \mathbb{R}^2 : 8/5 \leq x \leq 22/5 \}$

(2) (a) Symmetric 3-player game, $v(\{i\}) = 0$, $v_{ij} (\equiv v(\{i,j\})) = a$

$v_{123} = 3$, for what $a$ is the core non-empty?

Core $= \{ (x,y,z) \in \mathbb{R}^3 : x+y+z = 3,\ x+y \geq a,\ y+z \geq a,\ x+z \geq a,\ x,y,z \geq 0 \}$

$$\left.\begin{array}{l} a \leq x+y \leq 3 \\ a \leq y+z \leq 3 \\ a \leq z+x \leq 3 \end{array}\right\} \begin{array}{l} \Rightarrow 3a \leq 2(x+y+z) \leq 9 \\ \Rightarrow a \leq \frac{2}{3}(x+y+z) \leq 3 \end{array}$$

$\Rightarrow a \leq 2$ (& $a \geq 0$) $\Rightarrow \boxed{0 \leq a \leq 2}$

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(b) Symmetric 4-player game

$v_i = 0$, $v_{ij} = a$, $v_{ijk} = b$, $v(N) = 4$.

There are $\binom{4}{2}$ 2 player coalitions & $\binom{4}{3}$ 3 player coalitions.

for every 2-player coalitions $S = \{x_i, x_j\}$ $1 \le i, j \le 4$

$$a \le x_i + x_j \le b$$

$$\Rightarrow \binom{4}{2} a \le \binom{4-1}{2-1}\left(\sum_{i=1}^{N} x_i\right) \le \binom{4}{2} b$$

$$\cancel{6} a \le 3 \cdot 4 \le 6 \cdot b$$

$$0 \le a \le 2 \le b$$

Similarly: $\binom{4}{3} b \le \binom{4-1}{3-1}\left(\sum_{i=1}^{4} x_i\right) \le 4 \cdot \binom{4}{3}$

$$\cancel{4} b \le 3 \cdot \cancel{4} \le 4 \cdot \cancel{4} \Rightarrow \boxed{0 \le a \le 2 \le b \le 3}$$

(c) Finally, need to find conditions on $f(|S|) = v(S)$ for a symmetric game. Let $f(|S|) \equiv f_i$, $|S| = i$

$$v_i = f_1, \quad v_{ij} = f_2 \quad \dots, \quad v(N) = f_N$$

$$f_1 \le x_i \qquad \forall i$$

$$\Rightarrow N f_1 \le \sum x_i = f_N \Rightarrow f_1 \le \frac{1}{N} f_N$$

$$f_2 \le x_i + x_j \qquad \forall 1 \le i \ne j \le N$$

$$\binom{N}{2} f_2 \le \binom{N-1}{2-1} \cdot \sum x_i = (N-1) f_N \Rightarrow f_2 \le \frac{2}{N} \cdot f_N$$

In general, if $|S| = k$, $\binom{N}{k} f_k \le \binom{N-1}{k-1} \cdot f_N \Rightarrow \frac{N!}{k!(N-k)!} f_k \le \frac{(N-1)!}{(k-1)!(N-k)!} f_N$

$$\Rightarrow \boxed{f_k \le \frac{k}{N} f_N}$$

HW #9

1. Using Example 2 of §2.4,

$V_1 = V_2 = V_3 = V_{23} = 0$

$V_{12} = 30$, $V_{13} = V_{123} = 40$

Shapley value $\varphi(v)$ given by $\varphi = (\varphi_1, \varphi_2, \varphi_3)$

$$\varphi_i(v) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S|-1)!(n-|S|)!}{n!}\left[v(S) - v(S - \{i\})\right]$$

$$\varphi_1(v) = \frac{1}{3!}\Big[\underbrace{(1-1)!(3-1)!(0-0)}_{S=\{1\}} + \underbrace{(2-1)!(3-2)![30]}_{S=\{1,2\}} + \underbrace{(2-1)!(3-2)![40]}_{S=\{1,3\}}$$

$$+ (3-1)!(3-3)!(40)\Big]$$

$$= \frac{1}{3!}[0 + 30 + 40 + 2 \cdot 40] = \frac{150}{6} = 25$$

$$\varphi_2(v) = \frac{1}{3!}[30] = 5$$

$$\varphi_3(v) = \frac{1}{3!}[40 + 2 \cdot 10] = 10 \qquad \varphi = (25, 5, 10)$$

Core $= \{(x, 0, 40-x) : 30 \le x \le 40\} \Rightarrow \varphi \notin$ Core.

2 Find Shapley value of game with $v$: $v_\emptyset = 0$, $v_1 = 1$, $v_2 = 0$, $v_3 = -4$

$v_{12} = 2$, $v_{13} = -1$, $v_{23} = 3$, $v_N = 6$

$\varphi = (\varphi_1, \varphi_2, \varphi_3)$ where $\varphi_1(v) = \frac{1}{3!}\Big[(1-1)!(3-1)! \cdot 1 + (2-1)!(3-2)![2-0] + (2-1)!(3-2)![-1-1] + (3-1)!(3-3)!(6-3)\Big]$

$$\varphi_1(v) = \frac{1}{6}[\cancel{2\cdot 1} + 2 + (-2) + 2 \cdot 3] = 8/6$$

$$\varphi_2(v) = \frac{1}{6}\Big[(1-1)!(3-1)![\cancel{+1}0-0] + (2-1)!(3-2)!(1) + (2-1)!(3-2)!(3-(-4)) + (3-1)!(3-3)!(6+1)\Big] = \frac{22}{6}$$

$\varphi_3(v) = 1/6$