Homework 1

1. Give two examples of a $3 \times 3$ projection matrices that project onto one and two dimensional subspaces (thus four examples in all). Identify the subspaces.

2. Let $\mathbf{X}$ be a random $n$-vector and let $\mathbf{Y}$ be a random vector with $Y_1 = X_1$, $Y_i = X_i - X_{i-1}$, $i = 1, 2, \ldots, n$. Use matrix methods to solve the following:

   (a) If the $X_i$ are independent random variables, find the covariance matrix of $\mathbf{Y}$.
   
   \[ \Sigma_{XX} = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2) \text{ and } \mathbf{Y} = \mathbf{A} \mathbf{X} \text{ where } a_{ij} = 1 \text{ if } i = j, \]
   \[ a_{ij} = -1 \text{ if } i = j - 1 \text{ and } 0 \text{ otherwise.} \] (I also gave credit if it was assumed that $\Sigma_{XX} = \sigma^2 I$.) That is, $\mathbf{A}$ is a lower triangular matrix with the diagonal equal to 1 and the next lower diagonal equal to -1. $\Sigma_{YY} = \sigma^2 \mathbf{A} \Sigma_{XX} \mathbf{A}^T$ and
   
   \[
   A \Sigma_{XX} A^T = \begin{pmatrix}
   \sigma_1^2 & -\sigma_1^2 & 0 & 0 & \ldots & 0 \\
   -\sigma_1^2 & \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & \ldots & 0 \\
   0 & -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & \ldots & 0 \\
   \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & 0 & 0 & \ldots & \sigma_{n-1}^2 + \sigma_n^2
   \end{pmatrix}
   
   (b) If the $Y_i$ are independent random variables, find the covariance matrix of $\mathbf{X}$.
   
   Solve for $\mathbf{X}$ in terms of $\mathbf{Y}$ to find that $B = A^{-1}$ is a lower triangular matrix of 1’s. Then $B \Sigma_{YY} B^T$ has $ij$ element equal to $\sum_{k=1}^{\min(i,j)} \sigma_k^2$.

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with $E(X_i) = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let

   \[ Q = \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 \]

   Use matrix methods to find $E(Q)$ and show that $Q/2(n - 1)$ is an unbiased estimate of $\sigma^2$.

   Express $Q = ||A\mathbf{X}||^2 = X^T A^T \mathbf{A} \mathbf{X} = X^T B X$. $A$ is an $(n - 1) \times n$ matrix with $a_{ij} = -1$ if $i = j$, $a_{ij} = 1$ if $i = j - 1$ and 0 otherwise. $B$
is then a $n \times n$ matrix with $b_{11} = b_{nn} = 1$ and $b_{ii} = 2$ otherwise. Thus

$$E(Q(X)) = \mu^2 1^T B 1 + \text{trace}(B \Sigma_{XX})$$

$$= \mu^2 ||A1||^2 + \sigma^2 \text{trace}(B)$$

$$= 0 + 2(n - 1)\sigma^2$$