ON THE IDENTIFIABILITY OF
INFINITESIMAL GENERATORS

by

L. Le Cam*
University of California, Berkeley

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<u>1. Introduction</u>. Consider a Markov process Z with a finite number of states and stationary transition probabilities given by an infinitesimal generator A. The matrix A uniquely determines the matrix of transition probabilities P, given by $P_{j,k} = P[Z(1) = k | Z(0) = j]$. The matrix P can be written $P = \exp\{A\}$.

In many statistical problems A is unknown and must be estimated. This does not present much difficulty if enough independent replicates of the process are observed continuously in the entire interval $[0,1]$. However if instead one observes the process only at times 0 and 1 difficulties can occur. Indeed, in 1967, J. O. Speakman gave an example of two different generators A_1 and A_2 such that $\exp\{A_1\} = \exp\{A_2\}$. Thus, not only $\exp\{A_1\}$ and $\exp\{A_2\}$ might be very close for widely different A_j, they might be identical.

An isolated example is not enough to be a great source of concern, but we shall see that the phenomenon is widespread.

We shall consider only the case of a Markov process *with* 3 *states* and a *unique* invariant probability measure m such that $mP = m$. Furthermore we shall restrict ourselves to the case where P is diagonalizable, even though its eigenvalues are not all distinct.

This last restriction is not too serious: If $P = e^A$ for an A with distinct eigenvalues then P is diagonalizable whether its eigenvalues are distinct or not. Furthermore if $P = e^{A_1} = e^{A_2}$ for two different A_j then, one of the two A_j must have distinct eigenvalues.

Finally, we shall study only the case where m gives strictly positive probability to each of the three states. Other cases will be treated later.

Under these conditions we shall show the following:

If A has complex eigenvalues $\lambda_2 = -r + 2\pi ki$, $\lambda_3 = -r - 2\pi ki$ with k integer different from zero, then there is an A_0 with eigenvalues $(-r)$ and $(-r)$ such that $\exp\{A_0\} = \exp\{A\}$.

There may be other generators A_n with $\exp\{A_n\} = \exp\{A\}$ and with eigenvalues $\lambda_2 = -r + 2\pi ni$, $\lambda_3 = -r - 2\pi ni$ for some $n \neq k$. If so there may be a finite or continuum of them.

If A has complex eigenvalues $\lambda_2 = -r + ci$, $\lambda_3 = -r - ci$ for some $c \in (-\pi,\pi)$, there may be other generators A_n with $\exp\{A\} = \exp\{A_n\}$ and eigenvalues $\lambda_2 = -r + ci + 2\pi ni$, $\lambda_3 = -r + ci - 2\pi ni$, n integer different from zero. If so there are only a finite number of them.

<u>2. Conditions for indeterminacy of the generator</u>. Let A be an infinitesimal generator. Its first eigenvalue is zero, corresponding to the eigenvector $(1,1,1)'$. It has two other eigenvalues λ_2 and λ_3 that may be real, distinct or not. They may also be complex, in which case they are complex conjugate.

The corresponding eigenvalues of $P = e^A$ are always $1, e^{\lambda_2}, e^{\lambda_3}$. If v is an eigenvector of A with $Av = \lambda v$ then $Pv = e^{\lambda}v$.

Since $|e^{k\lambda}| \leq 1$ for all positive integers k, any eigenvalue of A must have a negative real part. Let q_2 and q_3 be the eigenvalues of P that are different from unity. Since $P = e^A$ one must have $q_j = \exp\{-r_j + ia_j\}$ for some real numbers r_j and q_j with $r_j > 0$. In particular $|q_j| = \exp\{-r_j\}$ so that r_j is well determined by $r_j = -\log|q_j|$.

Now to see whether $P = e^A$ determines A, consider the following cases.

CASE 1. $q_2 \neq q_3$ and they are real. Then if $\log|q_2| \neq \log|q_3|$ one must have $r_2 \neq r_3$. However since the complex eigenvalues of A must be conjugate, this implies $a_2 = a_3 = 0$. Hence the eigenvalues of A must be real $\lambda_2 = -r_2, \lambda_3 = -r_3$. If $q_2 \neq q_3$, real but $|q_2| = |q_3|$ the eigenvalues of A would be complex conjugate but differ by an odd multiple of π. Thus one would have, say $a_2 - a_3 \sim (2k+1)\pi$ and $a_3 = -a_2$. Equivalently $a_2 = (2k+1)\frac{\pi}{2}$, $a_3 = -(2k+1)\frac{\pi}{2}$. However $\exp\{(2k+1)\frac{\pi}{2}i\}$ is not real. Thus this is impossible.

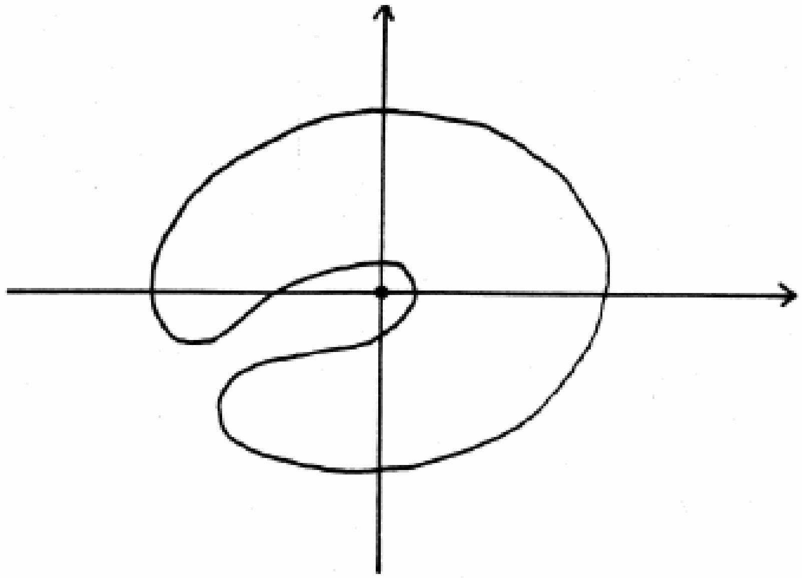
In summary $q_2 \neq q_3$, both real, implies that $\lambda_2 = -\log|q_2|$, $\lambda_3 = -\log|q_3|$ and in fact q_2 and q_3 are positive. The eigenvectors of P are well determined. They must be the same as the eigenvectors of

A. So A is well determined.

CASE 2. $q_2 \neq q_3$ and they are complex conjugate. Then $\lambda_2 = -r + ia + 2\pi ki$, $\lambda_3 = -r - ia - 2\pi ki$ for some integer k. The eigenvectors of A are the same as those of P and therefore determined (up to a multiplicative constant). Then, as we shall see, there may be several possibilities for k.

CASE 3. $q_2 = q_3$. Then they must be real and either of the form $q_2 = q_3 = e^{-r}$ or of the form $q_2 = q_3 = e^{-r \pm ri}$. In this case the eigenvectors of P form an entire two dimensional vector space. Thus neither the eigenvalues of A, nor the eigenvectors are well determined.

In all cases the exponential function admits a local inverse given by a Cauchy contour integral. The eigenvalues of P lie in the unit disk of the complex plane. If none of them lies on the negative part of the real line $\{\mathcal{Re}\, z \leq 0\, ,\ \mathrm{Im}\, z = 0\}$ (where $\mathcal{Re}$ means "real part" and Im means "imaginary part"), one can surround the eigenvalues by a contour C that stays strictly away from the negative real line. The principal branch of the logarithm, with $\log 1 = 0$, is analytic in a neighborhood of such a contour. Thus the standard Cauchy formula $B = \log P = \int_C (zI-P)^{-1} \log z\, dz$ gives a matrix such that $P = e^B$. This matrix B has eigenvalues $(0, -r+ic, -r-ic)$ for some $c \in (-\pi, \pi)$. One could attempt to extend the formula to the case where the eigenvalues of P are $(1, -e^{-r}, -e^{-r})$. This can be done by taking a contour that surrounds all the eigenvalues of P but does not wind around zero. A contour of the form



will do. However the Cauchy integral on such a contour gives a matrix B with eigenvalues $(0,-r+\pi i,-r+\pi i)$. This corresponds to a matrix with complex entries that cannot be an infinitesimal generator.

In other cases whether B given by the Cauchy formula is an infinitesimal generator will depend on the sign of the off diagonal entries of B. They must all be positive.

3. A bit of algebra. Let A be an infinitesimal generator with entries $a_{j,k}$. Assume that A is diagonalizable. Then it has a matrix of eigenvectors V of the type

$$V = \begin{pmatrix} 1 & v_{12} & v_{13} \\ 1 & v_{22} & v_{23} \\ 1 & v_{32} & v_{33} \end{pmatrix}$$

where the columns are eigenvectors and V is nonsingular. Thus one has $AV = V\Delta$ for a diagonal matrix Δ whose diagonal is $(0,\lambda_2,\lambda_3)$.

In all the cases where A is not well determined by P, there is an A with $P = e^A$ and with $\lambda_2 \neq \lambda_3$ and therefore $\lambda_3 = \overline{\lambda}_2$. For such a case it is convenient to write Δ as the diagonal matrix with entries $(0,-r,-r)$ to which is added a matrix $i\gamma J$

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

Let Π be the projection

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then Δ may be written

$$\Delta = -rI + r\Pi + i\gamma J .$$

This gives

$$A = V\Delta V^{-1} = -rI + rV\Pi V^{-1} + i\gamma VJV^{-1} .$$

The matrix

$$V\Pi V^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} V^{-1}$$

is a matrix whose rows are all identical to the first row of V^{-1}.

One has $PV = Ve^{\Delta}$. Also if m is the invariant measure with $m = mP$, one has

$$mPV = mV = MVe^{\Delta} .$$

If $V_{\circ k}$ is the k^{th} column of V this gives $mV_{\circ 1} = \sum_{j=1}^{3} m_j = 1$ and $mV_{\circ 2} = mV_{\circ 3} = 0$. Thus the first row of V^{-1} is equal to m.

This suggests looking for solutions of the form

$$B = -rI + rX\Pi X^{-1} + i\gamma XJX^{-1}$$

where

$$X = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 1 & x_{22} & x_{23} \\ 1 & x_{32} & x_{33} \end{pmatrix}$$

is an invertible matrix whose columns are eigenvectors of P.

PROPOSITION 1. *Assume that* P *is diagonalizable with two equal eigenvalues* $q_2 = q_3 = e^{-r}$. *Let* X *be a matrix of eigenvectors of* P *as above.*

Then

$$B_0 = -rI + rX\Pi X^{-1}$$

is an infinitesimal generator with $P = e^{B_0}$. *It is the unique infinitesimal generator* B *that satisfies* $P = e^B$ *with real eigenvalues.*

PROOF. It is easy to check that the rows of B_0 add up to zero. The off diagonal entries come from the matrix $X\Pi X^{-1}$ whose rows are all identical to the vector m given by the invariant probability measure. Thus B_0 is an infinitesimal generator. One can write $X^{-1}B_0X = -rI + r\Pi = \Delta$ where Δ is a diagonal matrix with diagonal $(0,-r,-r)$. Thus $\exp\{B_0\}$ is $Xe^{\Delta}X^{-1} = P$.

To show that it is the unique solution with real eigenvalues note that the Cauchy formula gives the unique solution with eigenvalues whose imaginary part is smaller than π in absolute value.

Note that the generator B_0 does not depend on the choice of the matrix X since $X\Pi X^{-1}$ always has rows equal to m. However if we take for X a *real* matrix, the formula $B = -rI + rX\Pi X^{-1} + i\gamma XJX^{-1}$ cannot be an infinitesimal generator unless γXJX^{-1} is zero. Thus we are led to seek other solutions where X is a complex matrix. If so and if X is the matrix of eigenvectors for a generator with distinct eigenvalues, the two vectors $X_{\circ 2}$ and $X_{\circ 3}$ must be complex conjugate and the matrix will take the form

$$\begin{pmatrix} 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \\ 1 & z_3 & \bar{z}_3 \end{pmatrix}$$

for three complex numbers z_j, $j = 1,2,3$. We shall now investigate what $IXJX^{-1}$ looks like for such a matrix. However it is easier to first compute the matrix $XJX^* = (\det X)XJX^{-1}$ where X^* is the matrix of cofactors of X.

Note that the rows of XJX^{-1} or XJX^* always add up to zero. Indeed $XJX^{-1}(|)$ is the first column of

$$XJX^{-1}X = XJ\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but

$$J\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \; .$$

So we shall not bother to computer the diagonal terms of XJX^{-1} except the first one.

Generally, if $X = \{x_{j,k}\}$, with a first column made of ones, one can write X^* in the form

$$\begin{matrix} x_{22}x_{33} - x_{23}x_{32} & x_{13}x_{32} - x_{12}x_{33} & x_{12}x_{23} - x_{13}x_{22} \\ x_{23} - x_{33} & x_{33} - x_{13} & x_{13} - x_{23} \\ x_{32} - x_{22} & x_{12} - x_{32} & x_{22} - x_{12} \end{matrix} \; .$$

For the matrix X given by the z_j this gives the following:

$$X^* = \begin{Bmatrix} 2i\,\mathrm{Im}\, z_2 z_3 & 2i\,\mathrm{Im}\, z_3\overline{z}_1 & 2i\,\mathrm{Im}\, z_1\overline{z}_2 \\ -(z_3 - z_2)^- & -(z_1 - z_3)^- & -(z_2 - z_1)^- \\ z_3 - z_2 & z_1 - z_2 & z_2 - z_1 \end{Bmatrix} .$$

In particular

$$\det X = 2i\,\mathrm{Im}[z_2\overline{z}_3 + z_3\overline{z}_1 + z_1\overline{z}_2] \; .$$

The matrix XJX^* is equal to $-2Q$ where Q is the real part of

$$\begin{pmatrix} -S_1 & \bar{z}_1(z_1-z_3) & \bar{z}_1(z_2-z_1) \\ \bar{z}_2(z_3-z_2) & -S_2 & \bar{z}_2(z_2-z_1) \\ \bar{z}_3(z_3-z_2) & \bar{z}_3(z_1-z_3) & -S3 \end{pmatrix}$$

where the term S_j is the sum of the off diagonal terms in the same row. The matrix $iXJX^{-1}$ is therefore $iXJX^{-1} = \frac{1}{K} Q$ where K is defined by $K = -\frac{1}{2i} \det X = \mathrm{Im}\{z_3\bar{z}_2 + z_1\bar{z}_3 + z_2z_1\}$.

Note that the rows of the matrix Q are organized as follows: One obtains $Q_{1,2}$ from $Q_{3,1}$ by changing the indices of the z_j according to the rotation scheme $1 \to 2 \to 3 \to 1$. Similarly $Q_{3,2}$ is obtained from $Q_{2,1}$ by the rotation. Also $Q_{1,3}$ comes from $Q_{3,2}$ and $Q_{2,3}$ comes from $Q_{1,2}$.

According to the same scheme $m_1 = \frac{1}{K} \mathrm{Im}\, z_3\bar{z}_2$ and m_2 and m_3 can be written by rotation of the indices. For simplicity in writing we shall denote the matrix $iXJX^{-1}$ by the letter Φ and let M be the matrix whose rows are identical to the invariant probability measure m. Then our presumed generator B takes the form

$$B = -rI + rM + \gamma\Phi .$$

It will be an infinitesimal generator (of something) if all the off diagonal terms of $rM + \gamma\Phi$ are positive.

To find out how these terms behave let us consider the terms $Q_{2,1}$ and $Q_{3,1}$. One can write $Q_{2,1} = \mathcal{R}e\, z_3\bar{z}_2 - |z_2|^2$ and $Q_{3,1} = |z_3|^2 - \mathcal{R}e\, z_3\bar{z}_2$. The other terms can be written by rotation of the indices. Note the following.

LEMMA 1. *The sum* $\sum_{j \neq k} Q_{j,k}$ *of the off diagonal terms of* Q *is equal to zero.*

This follows easily from the above formulas. However, it is important because it shows that the $Q_{j,k}$ cannot be all of the same sign.

Let $z_j = \rho_j e^{i\theta_j}$ and introduce the differences $\varphi_1 = \theta_3 - \theta_2$, $\varphi_2 = \theta_1 - \theta_3$, $\varphi_3 = \theta_2 - \theta_1$. Then $Q_{2,1}$ and $Q_{3,1}$ may be written

$$Q_{2,1} = \rho_2[\rho_3 \cos \varphi_1 - \rho_2]$$

$$Q_{3,1} = \rho_3[\rho_3 - \rho_2 \cos \varphi_1] .$$

Also $K = \rho_3\rho_2 \sin \varphi_1 + \rho_1\rho_3 \sin \varphi_2 + \rho_2\rho_1 \sin \varphi_3$.

According to previous remarks the vector m is the first row of X^{-1}. Thus one has $m_1 = \frac{\rho_3\rho_2}{K} \sin \varphi_1$ and the rest can be written by rotation. It would be more convenient to write that $m_j = C \frac{\sin \varphi_j}{\rho_j}$ or $\rho_j = C \frac{\sin \varphi_j}{m_j}$. The division is legal since the assumption, made throughout, that $m_j > 0$ for all j implies that $\rho_j > 0$ for all j. The entire system being homogeneous in the ρ_j we shall take

$$\rho_j = \frac{\sin \varphi_j}{m_j}$$

provided that $K > 0$. *This can be arranged by interchange of the last two columns of* X *and will be assumed in the sequel.*

With this choice for the ρ_j, the determinant term K becomes

$$K = \prod_{j=1}^{3} \frac{\sin \varphi_j}{m_j}$$

Similarly

$$Q_{2,1} = \frac{\sin \varphi_2}{m_2} [\frac{1}{m_3} \sin \varphi_3 \cos \varphi_1 - \frac{1}{m_2} \sin \varphi_2] ,$$

$$Q_{3,1} = \frac{\sin \varphi_3}{m_3} [\frac{1}{m_3} \sin \varphi_3 - \frac{1}{m_2} \sin \varphi_2 \cos \varphi_1] .$$

Now $\varphi_2 = -(\varphi_1 + \varphi_3)$. Thus $-\sin \varphi_2 = \sin \varphi_3 \cos \varphi_1 + \cos \varphi_3 \sin \varphi_1$ and the terms in Q become

$$Q_{2,1} = \frac{\sin \varphi_2}{m_2^2} \left\{ \left(\frac{m_3 + m_2}{m_3} \right) \sin \varphi_3 \cos \varphi_1 + \sin \varphi_1 \cos \varphi_3 \right\} ,$$

$$Q_{3,1} = - \frac{\sin \varphi_3}{m_3^2} \left\{ \left(\frac{m_3 + m_2}{m_2} \right) \sin \varphi_2 \cos \varphi_1 + \sin \varphi_1 \cos \varphi_2 \right\} .$$

For the matrix $\Phi = \frac{1}{K} Q$ this gives

$$\Phi_{2,1} = \frac{m_1 m_3}{m_2} \left\{ \frac{m_3 + m_2}{m_3} \operatorname{cotg} \varphi_1 + \operatorname{cotg} \varphi_3 \right\} ,$$

$$\Phi_{3,1} = - \frac{m_1 m_2}{m_3} \left\{ \frac{m_3 + m_2}{m_2} \operatorname{cotg} \varphi_1 + \operatorname{cotg} \varphi_2 \right\} .$$

The other off diagonal terms are obtained by rotation of the indices. They are

$$\Phi_{3,2} = \frac{m_2 m_1}{m_3} \left\{ \frac{m_1 + m_3}{m_1} \operatorname{cotg} \varphi_2 + \operatorname{cotg} \varphi_1 \right\} ,$$

$$\Phi_{1,2} = - \frac{m_2 m_3}{m_1} \left\{ \frac{m_1 + m_3}{m_3} \operatorname{cotg} \varphi_2 + \operatorname{cotg} \varphi_3 \right\}$$

$$\Phi_{1,3} = \frac{m_3 m_2}{m_1} \left\{ \frac{m_2 + m_1}{m_2} \operatorname{cotg} \varphi_3 + \operatorname{cotg} \varphi_2 \right\}$$

$$\Phi_{2,3} = - \frac{m_3 m_1}{m_2} \left\{ \frac{m_2 + m_1}{m_1} \operatorname{cotg} \varphi_3 + \operatorname{cotg} \varphi_1 \right\} .$$

We have written them all out in full to show that they cannot all vanish. If for instance one has $\Phi_{3,1} = \Phi_{3,2} = 0$ then

$$\cot g\, \varphi_2 = -\frac{m_3+m_2}{m_2} \cot g\, \varphi_1 = \left(\frac{m_3+m_2}{m_2}\right)\left(\frac{m_1+m_3}{m_1}\right) \cot g\, \varphi_2 .$$

This is an impossibility unless $\cot g\, \varphi_2 = 0$ or $\cot g\, \varphi_2 = \pm\infty$. Now $\cot g\, \varphi_2 = 0$ means that $\varphi_2 = \frac{\pi}{2}$ (mod π). If this holds for all three φ_j we have $\varphi_1 + \varphi_2 + \varphi_3 = 0 = \frac{3\pi}{2}$, (mod π). This is a contradiction. The possibility $\cot g\, \varphi_j = \pm\infty$ is ruled out by the condition that $K = \prod_j \left(\frac{\sin \varphi_j}{m_j}\right)$ is not zero.

Now let us look at other conditions that must be satisfied by the angles φ_j. One of them is the orthogonality relation $\sum_j m_j z_j = 0$. Taking into account the fact that the vector m is proportional to the first row of X^*, this relation becomes

$$z_1 \operatorname{Im} \bar{z}_3 z_2 + z_2 \operatorname{Im} z_1 \bar{z}_3 + z_3 \operatorname{Im} z_2 \bar{z}_1 = 0 .$$

Equivalently

$$\sum_j e^{i\theta_j} \sin \varphi_j = 0 .$$

Dividing by $e^{i\theta_j}$, one obtains

$$\sin \varphi_1 + e^{i\varphi_3} \sin \varphi_2 + e^{-i\varphi_2} \sin \varphi_3 .$$

It is easily verifiable that this is always equal to zero.

We shall now look at the consequences of this state of affairs for the possible multiplicity of the solutions of $e^A = P$.

<u>4. The range of indeterminacy of the solutions of</u> $e^A = P$. Let us consider a transition matrix P with eigenvalues $(1, e^{-r+ia}, e^{-r-ia})$ and a matrix of eigenvectors $X = (1, Z, \overline{Z})$ where $Z' = (z_1, z_2, z_3)$ is a vector of complex numbers written $z_j = \frac{\sin \varphi_j}{m_j} e^{i\theta_j}$ as in Section 3. Let Φ be the matrix $iXJX^{-1}$ and let M be the matrix whose rows are all equal to the invariant probability measure m such that $mP = m$.

According to Section 3, a necessary and sufficient condition for $B = -rI + rM + \gamma\Phi$ to be the infinitesimal generator of some process is that all the off diagonal terms of $rM + \gamma\Phi$ be nonnegative.

Note that this condition involves only the angles φ_j. This is as it should be since one could multiply Z by any one of the $e^{-i\theta_j}$.

This gives the following result. Let $\theta_1, \theta_2, \theta_3$ be three angles such that $|\theta_j| \leq \pi$. Let $\varphi_1 = \theta_3 - \theta_2$, $\varphi_2 = \theta_1 - \theta_3$ and $\varphi_3 = \theta_2 - \theta_1$. Let $Z' = (z_1, z_2, z_3)$ be given by $z_j = \frac{\sin \varphi_j}{m_j} e^{i\theta_j}$ and form the matrices M and Φ as described in Section 3.

THEOREM 1. *Assume* $\prod_j \sin \varphi_j \neq 0$. *In order that* $B = -rI + rM + \gamma\Phi$ *be the infinitesimal generator of some Markov process it is necessary and sufficient that all the off diagonal terms of* $rM + \gamma\Phi$ *be nonnegative. If so* B *will generate* P *by the formula* $P = e^B$ *if and only if the following conditions are satisfied.*

i) $mP = m$

ii) *The eigenvalues of* P *are* $(1, e^{-r+i\gamma}, e^{-r-i\gamma})$.

iii) *The vector* Z *is an eigenvector of* P.

If the eigenvalues of P *are real (and therefore equal, by* (ii) *above), the condition* (iii) *is automatically satisfied.*

PROOF. This follows immediately from the relations given in

Sections 2 and 3.

COROLLARY. *If* P *admits complex eigenvectors and an invariant measure* m *with* $m_j > 0$ *for all* j, *then for all sufficiently large values of* t *the equation* $P^t = e^{tA}$ *will have multiple solutions.*

PROOF. Consider the matrix X of eigenvectors of P and form $B = -rI + rM + \beta\Phi$. This will be an infinitesimal generator as soon as $\inf_j rm_j \geq \sup_{j \neq k} |\beta\Phi_{j,k}|$. One can write $\exp\{tB\} = Xe^{t\Delta}X^{-1}$ where Δ is the diagonal matrix with diagonal entries $(0, -r+i\beta, -r-i\beta)$. Let $(1, e^{-r+i\gamma}, e^{-r-i\gamma})$ be the eigenvalues of P with $|\gamma| \leq \pi$. Then one will have $P^t = e^{tB}$ whenever $t\gamma = t\beta$, mod 2π. Now β is allowed to vary in some interval $[-a,a]$ with $a = (\inf_j rm_j)[\sup_{j \neq k} |\Phi_{j,k}|]^{-1}$. As soon as $at \geq 4\pi$, the relation $t\gamma = t\beta \bmod 2\pi$ will have several solutions, the number of solutions increasing as t increases.

In the above argument we have assumed $|\gamma| \leq \pi$, but one can assume that $\gamma \in (-\pi, \pi]$ for definiteness. Then there are three possible cases: (1) $\gamma \neq 0$, $\gamma \neq \pi$, (2) $\gamma = 0$ and (3) $\gamma = \pi$. In the first case, the eigenvector Z is well determined. Thus, there is only one matrix Φ. Therefore in such a situation the number of solutions of $e^{tB} = P^t$ is finite for each given t. That number increases roughly linearly as t increases.

In the cases where $\gamma = 0$ the situation is entirely different. *Every choice of vector* Z *such that* $\prod_j \sin \varphi_j \neq 0$ *will give a possible matrix* Φ *and* $B = -rI + rM + 2\pi n\Phi$ *will be a generator such that* $P = e^B$ *provided that the off diagonal terms* $rM_{j,k} + 2\pi n\Phi_{j,k}$ *be nonnegative.*

Similarly, if $\gamma = \pi$, the matrices $B = -rI + rM + (2n+1)\pi\Phi$ will be such that $P = e^B$ and they will be generators if they satisfy the appropriate positivity requirements. This leads to the following result.

PROPOSITION 2. *Let* P *be fixed with* $\gamma = 0$ *and let* $\varphi^\circ = (\varphi_1^\circ, \varphi_2^\circ, \varphi_3^\circ)$ *be such that, for the corresponding matrix* Φ°, *the off diagonal entries of* $B = -rI + rM + 2\pi n\Phi^\circ$, *are all positive for a given positive integer* n. *Then, for any integer* m *such that* $0 < m < n$ *there is a neighborhood* V *of* φ° *such that if* Φ *corresponds to a* $\varphi \in V$ *then* $B = -rI + rM + 2\pi m\Phi$ *is an infinitesimal generator with* $P = e^B$.

PROOF. The determinant $\prod_j \frac{\sin \varphi_j}{m_j}$ is a continuous function of φ. Since it does not vanish at φ°, its absolute value is bounded away from zero in a neighborhood of φ°. In that neighborhood, Φ is a continuous function of the vector of angles φ. By assumption the terms $rM_{j,k} + 2\pi n\Phi^\circ_{j,k}$ are all nonnegative. Consider a pair j, k where $\Phi^\circ_{j,k} < 0$. Then $\Phi_{j,k} < 0$ and $|\Phi_{j,k}| < \frac{n}{m} |\Phi^\circ_{j,k}|$ for φ in some neighborhood of φ°. Therefore $rM_{j,k} + 2\pi m\Phi_{j,k} \geq 0$ for such a neighborhood. If $rM_{j,k} + 2\pi n\Phi^\circ_{j,k} \geq 0$ by virtue of the fact that $\Phi^\circ_{j,k} = 0$, then $|\Phi_{j,k}|$ will remain small for φ in some neighborhood of φ°. Since $rM_{j,k} = rm_k > 0$, the combination $rM_{j,k} + 2\pi m\Phi_{j,k}$ will still be positive in a neighborhood of φ°. Hence the assertion.

A similar argument can be carried out for the case where $\gamma = \pi$.

In summary, for a given φ and $B = -rI + rM + \beta\Phi$ the equation $P = e^B$ can have only a finite number of solutions. Thus if $\gamma \neq 0$, $\gamma \neq \pi$, the equation $P = e^B$ can have only a finite number of solutions. If however $\gamma = 0$ or $\gamma = \pi$, one may be able to obtain an infinity of solutions by varying φ. For the case $\gamma = 0$ this will happen whenever $-rI + rM + 4\pi\Phi^\circ$ is an infinitesimal generator, and this will certainly happen whenever r is sufficiently large.

Increasing r amounts to the same thing as increasing the length of observation $[0,1]$ to $[0,t]$, $t > 1$. Therefore, if $\gamma = 0$ there will be integer times t where $P^t = e^{tB}$ has a set of distinct solutions having the power of the continuum.

To terminate, let us note that two generators A and B that yield the same P can correspond to very different behaviors of the Markov process they define. For the interval $[0,1]$, the example given by J. O. Speakman uses the generators

$$A = \frac{4n\pi}{\sqrt{3}}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$B = \frac{4n\pi}{\sqrt{3}}\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}$$

where n is a positive integer.

For the generator A the process goes through the states circularly $1 \to 2 \to 3 \to 1$. For the generator B the process goes from one state to the other two selecting each one of them with probability $\frac{1}{2}$. One could note that B is the unique generator with real eigenvalues corresponding to $P = e^A$. Also if one takes for n an integer ≥ 2, the equation $P = e^G$ has a continuum of solutions. The Speakman example can be used to show indeterminacy for a case where P has a pair of distinct complex conjugate roots. For the matrix A, the matrix Φ is

$$\Phi = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

For $n = 1$, the eigenvalues are $(0,-r+2\pi i,-r-2\pi i)$ with $r = 2\pi\sqrt{3}$. Consider then the matrix

$$D = \pi\sqrt{3}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

It has the same matrix Φ. Its eigenvalues are $(0,-a+\frac{3\pi}{2}i,-a-\frac{3\pi}{2}i)$ with $a = \frac{3\pi\sqrt{3}}{2}$. However, modulo 2π, the eigenvalues are equivalent to $(0,-a-\frac{\pi}{2}i,-a+\frac{\pi}{2}i)$. Thus the generators $D^{(1)} = -aI + aM + \frac{3\pi}{2}\Phi$ and $D^{(2)} = -aI + aM - \frac{\pi}{2}\Phi$ will yield the same transition matrix P with eigenvalues

$$\left(1, e^{-a-\frac{\pi i}{2}}, e^{-a+\frac{\pi i}{2}}\right).$$

The generator $D^{(2)}$ is what one obtains by applying the standard Cauchy formula to P.

5. Conclusions. According to Theorem 1 and its corollaries, it can easily happen that a transition matrix P is insufficient for the determination of the infinitesimal generator of the Markov process that gave P. This can happen even if P has distinct eigenvalues and well determined eigenvectors. However it cannot happen if the eigenvalues of P are real and distinct. If they are complex and distinct, multiplicities can occur only if the modulus of those different from unity is sufficiently small corresponding to a sufficiently large value of the number called $r = -\log|q_2|$ in the notation of Section 2. If faced with a particular P one can determine the invariant measure m. If in addition the eigenvectors are determined then one can construct the matrix Φ and check whether multiple solutions are possible. If on the contrary the second and third eigenvalues of P are equal, one is faced with the problem that all vectors $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ with $\varphi_1 + \varphi_2 + \varphi_3 = 0$ are admissible in the construction of Φ. However note that, for a given invariant measure m, $\sup_{j \neq k} \Phi^+_{j,k}$ and $\sup_{j \neq k} \Phi^-_{j,k}$ admit a lower bound independent of φ. Indeed, as argued in Section 3, $\Phi_{3,1}$ and $\Phi_{3,2}$ cannot vanish simultaneously. Neither can $\Phi_{1,2}$ and $\Phi_{1,3}$ or $\Phi_{2,1}$ and $\Phi_{2,3}$. Thus, as long at $\prod_j \sin \varphi_j \geq \alpha > 0$ for a fixed α $\inf_\varphi \sup_{j \neq k} \Phi^-_{j,k}$ stays bounded away from zero. It is also easy to check that if any φ_j tends to zero then $\sum_{j \neq k} \Phi^-_{j,k}$ must tend to infinity. Therefore there is some number $\varepsilon(m) > 0$ such that $\inf_\varphi \sup_{j \neq k} \Phi^-_{j,k} \geq \varepsilon(m)$. This again shows that multiplicities will not occur unless r is sufficiently large. For this conclusion, it is easier to work with $\frac{1}{m_k} \Phi_{j,k}$ instead of $\Phi_{j,k}$ itself. If $\eta(m) = \inf_\varphi \sup_{j \neq k} \frac{1}{m_k} \Phi^-_{j,k}$, then in order for multiplicities to occur one must have $r \geq 2\pi\eta(m)$.

According to the above, if r is sufficiently small, or if the eigenvalues of P are real and distinct, the generator A such that $P = e^A$ is well determined. This means that if considering two possibilities say $P^{(1)}$ and $P^{(2)}$ for P, one could hope to obtain lower bounds for the information available when observing at zero and one only in terms of the information available by continuous observation in $[0,1]$. However such lower bounds do not seem to have been derived in the literature.

REFERENCES

SPEAKMAN, J. O. (1967). "Two Markov chains with a common skeleton. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*. 7 224.