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NOTE ON A CERTAIN CLASS OF MEASURES

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1. Introduction. Let $(\mathcal{X}, \mathcal{U})$ be a uniform space and let Δ be the space of bounded uniformly continuous real functions defined on $\mathcal{X}$. For the uniform norm this is a Banach space with dual Δ^*. The present note is devoted to an elaboration of the properties of a certain subspace $\mathcal{M}_u$ of Δ^* which seems to reflect the properties of the structure $\mathcal{U}$ more reasonably than the usual spaces of measures. The space $\mathcal{M}_u$ arises naturally in various arguments about convolutions or Fourier transforms on linear space. For simplicity we shall call a subset S of Δ a U E B set if it is uniformaly equicontinuous and bounded. Uniform convergence on the U E B sets of Δ defines on Δ^* a uniform structure which will be denoted [U]. On Δ or subsets of Δ the structure of uniform convergence on the precompact subsets of $\mathcal{X}$ will be called simply the precompact convergence. With this notation we can define the

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This paper was written with partial support of the National Science Foundation Grant No. GP-8690 and 15283.

following subspaces of Δ^*

1) The space $\mathcal{M}_s$ of linear functionals with finite support on $\mathcal{X}$. Specifically $\varphi \in \mathcal{M}_s$ if $\langle\varphi,\gamma\rangle = \Sigma c_j\gamma(x_j)$ for a finite set of points $\{x_j\}$ and for real numbers c_j.

2) The space $\mathcal{M}_p$ of linear functionals whose restrictions to the unit ball $B = \{\gamma \; ; \; \gamma \in \Lambda, \; |\gamma| < 1\}$ are continuous for the precompact convergence.

3) The space $\mathcal{M}_u$ closure of $\mathcal{M}_s$ in Δ^* for the [U] structure.

4) The space $\overline{\mathcal{M}}_u$ of elements of Δ^* whose restrictions to U E B subsets of Δ are continuous for the precompact convergence.

It is almost immediate that

$$\mathcal{M}_s \subset \mathcal{M}_p \subset \mathcal{M}_u \subset \overline{\mathcal{M}}_u \subset \Delta^*$$

We shall show below that $\mathcal{M}_u$ and $\overline{\mathcal{M}}_u$ are always identical and that $\mathcal{M}_p = \mathcal{M}_u$ if $\{\mathcal{X}, \}$ is metrizable. It will also be shown that the balls of $\mathcal{M}_u$ are complete for [U] so that the weak compactness criteria of Grothendieck [1] may be applied to $\mathcal{M}_u$ for the topology $w[\mathcal{M}_u, \Delta]$. The positive cone $\mathcal{M}_u^*$ of $\mathcal{M}_u$ has the remarkable property that on it the topologies defined by $w[\mathcal{M}_u,\Delta]$ and by [U] coincide.

The third section of this note elaborates a characterisation of $\mathcal{M}_u$ through maps of $\mathcal{X}$ into metric spaces. This is refined in section 4, where results analogus to those stated for linear spaces in [2] are proved. Section 5 gives some results relative to the case where Δ happens to be the space of all bounded continuous functions for the topology induced by $\mathcal{U}$ and gives in this case a characterisation of $\mathcal{M}_u$ through properties of images on paracompact spaces. In this case the result obtained appears to be a modification of some results of E. Granirer [3].

2. Basic properties of the space $\mathcal{M}_u$. With the terminology of the introduction, let S be a U E B subset of Δ and let $\bar{S}$ be its pointwise closure in the space $\mathcal{F}(\mathcal{X},R)$ of all real functions on $\mathcal{X}$. This set $\bar{S}$ is also a U E B subset of Δ. The same statement applies to the convex hull of S. For the precompact convergence $\bar{S}$ is a compact set. Let $\mathcal{S}$ be the family of all convex symmetric U E B subsets of Δ which are compact for the precompact convergence.

It is easily seen that the original structure $\mathcal{U}$ on $\mathcal{X}$ is precisely the same as the structure [U] of uniform convergence on the elements of $\mathcal{S}$. Indeed the very definition of U E B sets implies that [U] is weaker than $\mathcal{U}$ on $\mathcal{X}$. However $\mathcal{U}$ may be defined by a family of pseudo-metrics $(x,y) \to \rho(x,y)$ such that $0 \leq \rho(x,y) \leq 1$. For each such ρ let S_ρ be the set of functions $x \to \rho(x,y)$ indexed by

$y \in \mathcal{X}$. The set $\mathcal{E}_\rho$ is an U E B set and $\rho(x_1,x_2) = \sup\{|\gamma(x_1) - \gamma(x_2)| ; \gamma \in S_\rho\}$. In fact among structures of uniform convergence on families of subsets of Δ, the structure [U] is strongest which agrees with $\mathcal{U}$ or $\mathcal{X}$.

<u>Lemma 1</u>. <u>Let</u> S <u>be an</u> U E B <u>set which is compact for the topology of pointwise convergence on</u> $\mathcal{X}$. <u>Then</u> S <u>is also compact for</u> $w[\Delta, \overline{\mathcal{M}}_u]$.

<u>Proof</u> By definition the elements of $\overline{\mathcal{M}}_u$ are continuous for the precompact convergence but on S this coincides with pointwise convergence. The result follows.

It is clear from the definition that both $\mathcal{M}_u$ and $\overline{\mathcal{M}}_u$ are closed subspaces of Δ^* for the structure [U]. Thus they are also closed for the stronger topology defined by the norm of Δ^*. The space $\mathcal{M}_p$ is also closed for the norm. Note also the following.

<u>Lemma 2</u>. <u>Let</u> μ <u>be an element of</u> $\mathcal{M}_u$. <u>Then its positive part</u> μ^+ <u>belongs to</u> $\mathcal{M}_u$. <u>The same statement applies also to</u> $\mathcal{M}_p$ <u>and</u> $\overline{\mathcal{M}}_u$.

<u>Proof</u>/ One can define μ^+ by the relation $\langle\mu^+,\gamma\rangle = \sup\{\langle\mu,u\rangle ; 0 \le u \le \gamma, u \in \Delta\}$ for any γ in the positive cone Δ^+ of Δ. Let $v \in \Delta$ be such that $0 \le v \le 1$ and such that $\langle\mu^+,1\rangle \le \langle\mu,v\rangle + \varepsilon = \langle\mu^+,v\rangle - \langle\mu^-,v\rangle + \varepsilon$. This gives $\langle\mu^+,(1-v)\rangle + \langle\mu^-,v\rangle \le \varepsilon$. For any $\gamma \ge 0$, $\gamma \in \Delta$ we can write

$$\langle\mu^+,\gamma\rangle = \langle\mu^+, v\gamma\rangle - \langle\mu^-, v\gamma\rangle + \langle\mu^+, v\gamma\rangle + \langle\mu^+, (1-v)\gamma\rangle$$

$$= \langle\mu, v\gamma\rangle + \langle\mu^-, v\gamma\rangle + \langle\mu^+, (1-v)\gamma\rangle$$

$$\leq \langle\mu, v\gamma\rangle + \varepsilon\ ||\gamma||$$

Define the functional $v\cdot\mu$ by $\langle v\cdot\mu,\gamma\rangle = \langle\mu, v\gamma\rangle$. The foregoing inequality gives $||\mu^+ - v\cdot\mu|| \leq 2\,\varepsilon$. If S is an U E B set so is the set $\{v\gamma\ ;\ \gamma \in S\}$. Thus $v\cdot\mu \in \mathcal{M}_u$ whenever μ does. Similarly if $\mu \in \mathcal{M}_p$ (resp $\overline{\mathcal{M}_u}$) then $v\cdot\mu \in \mathcal{M}_p$, (resp $\overline{\mathcal{M}_u}$). It follows that μ^+ limit for the norm topology is also in the appropriate space.

<u>Lemma 3</u>. <u>The spaces</u> $\mathcal{M}_p$, $\mathcal{M}_u$ <u>and</u> $\overline{\mathcal{M}_u}$ <u>are bands in</u> Δ^* .

<u>Proof</u> Since these spaces are closed in Δ for the norm topology and since lemma 2 applies to them, it is sufficient to show that if $0 \leq \nu \leq \mu \in \mathcal{M}_p$ then $\nu \in \mathcal{M}_p$ and similarly for $\mathcal{M}_u$ and $\overline{\mathcal{M}_u}$. This can be done exactly as in lemma 2 using the fact that $||\nu - v\cdot\mu|| < \varepsilon$ for some $v \in \Delta$. For this latter fact see [4], [5] and [6] .

<u>Proposition 1</u>. <u>The positive cone</u> $\mathcal{M}_u^+$ of $\overline{\mathcal{M}_u}$ <u>is the</u> [U] <u>closure in</u> Δ^* <u>of the positive cone</u> $\mathcal{M}_s^+$ of $\mathcal{M}_s$. <u>The spaces</u> $\mathcal{M}_u$ <u>and</u> $\overline{\mathcal{M}_u}$ <u>coincide</u>. <u>The balls</u> $\{\mu\ ;\ ||\mu|| \leq b\}$ <u>of</u> $\mathcal{M}_u$ <u>are complete for</u> [U].

<u>Proof</u> Let $C = \mathcal{M}_s^+$ and let $\overline{C}$ be its [U] closure in $\overline{\mathcal{M}}_u$. Let μ be an element of $\overline{\mathcal{M}}_u^+$. Suppose that $\mu \notin \overline{C}$. Then there is a [U] continuous linear functional γ or $\mathcal{M}_u$ and numbers $a < a + \varepsilon$ such that

$$\langle \mu, \gamma \rangle = a < a + \varepsilon \leq \langle \varphi, \gamma \rangle$$

for all $\varphi \in C$. According to lemma 1 the functional γ is in fact given by some element of Δ. The inequality $a + \varepsilon \leq \langle \varphi, \gamma \rangle$ for $\varphi \in C$ implies $\gamma \geq 0$ hence $a + \varepsilon \leq \inf_{\varphi \in C} \langle \varphi, \gamma \rangle = 0$. This gives $\langle \mu, \gamma \rangle = a < 0$ contrary to the assumption $\mu \geq 0$.

It follows from this that $\mathcal{M}_u \supset \overline{\mathcal{M}}_u^+$ and therefore $\mathcal{M}_u \supset \overline{\mathcal{M}}_u^+ - \overline{\mathcal{M}}_u^+ = \overline{\mathcal{M}}_u$. Since $\mathcal{M}_u \subset \overline{\mathcal{M}}_u$ the second statement is proved. For the third statement let $V = \{\mu ; ||\mu|| \leq b ; \mu \in \mathcal{M}_u\}$. If $\mu \in V$ then $\mu^+ \in V$ and $\mu^- \in V$. However by the first part of the argument μ^+ is a [U] limit of elements $\varphi \in \mathcal{M}_s^+$ such that $||\varphi|| \leq ||\mu^+||$ a similar statement applies to μ^-. Hence V is the [U] closure of $\{\mu ; ||\mu|| \leq b ; \mu \in {}_s\}$ in Δ^*. Since the corresponding ball of Δ^* is $w[\Delta^*, \Delta]$ compact and therefore [U] complete the proposition follows. <u>Note</u> - The preceding argument implies also that $\mathcal{M}_u^+$ is complete for [U]. We do not know whether this is true for $\mathcal{M}_u$ itself.

The completeness statement included in proposition 1 allows the application of the $w[\mathcal{M}_u, \Delta]$ compactness criteria given by

Grothendieck [1]. In particular one can obtain the following proposition in which the space (m) is the usual Banach space of sequences of real numbers with its uniform norm and (m^*) is the dual of (m).

Corollary 1 Let A be a norm bounded subset of $\mathcal{M}_u$. Topologize the U E B subsets of Δ by the precompact convergence. The following statements are all equivalent

1) A is relatively compact in $\mathcal{M}_u$ for $w[\mathcal{M}_u, \Delta]$

2) For each compact U E B set S the restrictions of A to S form a relatively compact set in C(S) for the topology of pointwise convergence on S

3) For every U E B sequence $\{\gamma_n\}$, $\gamma_n \in \Delta$ and every sequence $\{\mu_k\}$, $\mu_k \in A$ one has $\lim\limits_n \lim\limits_k \langle\mu_k, \gamma_n\rangle = \lim\limits_k \lim\limits_n \langle\mu_k, \gamma_n\rangle$ provided that all the limits in these expressions exist.

4) For every [U] continuous linear map of $\mathcal{M}_u$ into (m) the image of A is $w[(m),(m^*)]$ relatively compact in (m)

5) For every uniformly continuous map $x \rightsquigarrow (x)$ of $(\mathcal{X}, U)$ into (m) the corresponding image of A on $\Delta^*(m)$ is relatively compact in the space $\mathcal{M}_u[(m)]$ defined by (m).

The equivalence of (1)-(3) is a restatement in this particular case of theorem 7 of [1]. The statements (4) and (5) arise as follows. Let $\{\gamma_n\}$ be a U E B sequence. To each $\mu \in \Delta$ assign the element $s(\mu)$ of (m) whose nth coordinate is $\langle\mu,\gamma_n\rangle$. This is obviously a [U] continuous map of $\mathcal{M}_u$ into (m). Its adjoint transform (m^*) into Δ. Hence $\mu \rightsquigarrow s(\mu)$ transform $w[\mathcal{M}_u,\Delta]$ compact sets of $\mathcal{M}_u$ into $w[(m),(m^+)]$ compact sets. However a $w[(m),(m^*)]$ compact set satisfies the double limit condition of statement (3). For statement (5) the argument is similar except that one considers the map $x \rightsquigarrow s(x)$ with nth coordinate of $s(x)$ equal to $\gamma_n(x)$.

One might also note that in statement (3) it is not necessary to check the iterated limit condition for all U E B sequences $\{\gamma_n\}$ but only for a family $\mathcal{F}$ of U E B sequences large enough so that any U E B sequence be contained in the closed symmetric convex hull of some sequence belonging to the family $\mathcal{F}$.

Corollary 2 *Let* A *be a norm bounded subset of* $\mathcal{M}_u$. *This set is relatively compact in* $\mathcal{M}_u$ *for the structure* [U] *if and only if for each compact* U E B *set* S *the restrictions of* A *to* S *form an equicontinuous subset of* C(S).

This is an immediate consequence of Ascoli's theorem. It is mentioned here primarily for purposes of comparison with the "tightness" statement of proposition 4 below.

Proposition 2 *Let* μ *be an element of the positive cone* $(\Delta^*)^+$

of Δ^*. The condition $\mu \in \mathcal{M}_u^+$ is equivalent to the statement that every filter on $(\Delta^*)^+$ which converges to μ for $w[\Delta^*,\Delta]$ already converges to μ for the structure [U].

Proof Every element of $(\Delta^*)^+$ is a $w[\Delta^*,\Delta]$ limit of a filter of positive measures with finite support (that is $(\Delta^*)^+$ is the $w[\Delta^*,\Delta]$ closure of $\mathcal{M}_s^+$). Hence the condition $\mu \in \mathcal{M}_u^+$ is certainly implied by the other. Conversely let $\mu \in \mathcal{M}_u^+$, $||\mu|| = 1$ and let $\mathcal{F}$ be a filter on $(\Delta^*)^+$ which converges to μ for $w[\Delta^*,\Delta]$. Let S be a U E B subset of Δ. Let $\rho(x,y) = \sup\left\{|\gamma(x) - \gamma(y)| \; ; \; \gamma \in S\right\}$ and let S_m be the set of functions γ such that $|\gamma| \leq m$ and $|\gamma(x) - \gamma(y)| \leq m\, \rho(x,y)$. This set S_m is also a U E B subset of Δ. One can assume that $\gamma \in S$ implies $||\gamma|| \leq 1$.

Since $\mu \in \mathcal{M}_u$ for each $\varepsilon > 0$ there is a $\nu \geq 0$ with finite support F such that $||\nu|| = 1$ and $|\langle\mu,\gamma\rangle - \langle\nu,\gamma\rangle| < \frac{\varepsilon}{8}$ for every $\gamma \in S_m$. Let G be the set $G = \{x \; ; \; \rho(x,F) > \frac{1}{m}\}$. Define a function u by $u(x) = \rho(x,G)\,[\rho(x,F) + \rho(x,G)]^{-1}$. It can be checked that $u \in 4\, S_m$. Since by construction $u(x) = 1$ for $x \in F$ we have $\langle\nu,u\rangle = 1$ and therefore $\langle\mu,u\rangle \geq 1 - \varepsilon/2$. Thus there is a set A of the filter $\mathcal{F}$ such that $\varphi \in A$ implies $\langle\varphi,u\rangle \geq 1 - \varepsilon$.

Let $x_1, x_2, \cdots, x_n$ be the elements of F. Dividing the range of values of the functions $\gamma \in S_1$ onto intervals of length at most $\frac{1}{m}$ one can find functions $\gamma_j \in S_1, j=1,2,\cdots,N,$

with $N \leq (2m)^n$ such that $\inf\limits_j \sup\limits_i |\gamma(x_i) - \gamma_j(x_i)| \leq \frac{1}{m}$ for every $\gamma \in S_1$. This gives immediately

$$\inf_j ||u\gamma - u\gamma_j|| \leq \frac{1}{m} + \frac{2}{m}$$

for every $\gamma \in S_1$ according to the definition of ρ itself. By assumption $\lim\limits_{\mathcal{F}} \langle\varphi, u\gamma_i\rangle = \langle\mu, u\gamma_i\rangle$ for each j. Hence

$$\lim_{\mathcal{F}} \ \sup \ \sup_{\gamma\in S_1} |\langle\varphi, u\gamma\rangle - \langle\mu, u\gamma\rangle| \geq \frac{3}{m} \ .$$

and finally

$$\lim \ \sup \ \sup_{\gamma\in S} | \ \varphi, u\gamma \ - \ \mu, u\gamma \ | \leq \frac{3}{m} + \varepsilon \ .$$

Since ε and m are arbitrary the result is proved.

<u>Corollary</u> <u>On the cone</u> $\mathcal{M}_u^+$ <u>the topology indexed by</u> $[U]$ <u>is the same as the</u> $w[\mathcal{M}_u, \Delta]$ <u>topology</u>.

3. <u>Images of</u> $\mathcal{M}_u$ <u>on metric spaces</u>.

As noted before, given a uniform space $(\mathcal{X}, \mathcal{U})$ there is a natural correspondence between the bounded pseudometrics defining $\mathcal{U}$ and the pseudometrics $\rho(x,y) = \sup\{|\gamma(x) - \gamma(y)|; \gamma \in S\}$ for sets S which are U E B subsets of Δ.

Any pseudometric ρ can be used to define a set

$\Lambda_\rho(a,b)$ functions γ such that $|\gamma| \leq a$ and $|\gamma(x) - \gamma(y)| \leq b\,\rho(x,y)$. Such a pseudometric induces pseudo-norm on Δ^* by the relation

$$||\mu||\rho = \sup\left\{|\langle\mu,\gamma\rangle| \ ; \ \gamma \in \Lambda\,\rho(1,1)\right\} .$$

We shall call this the Dudley psudo-norm associated with the pseudo-metric ρ. (see [7]).

Let F_i $i = 1,2$ be two subsets of $\mathcal{X}$. Define $\rho(x,F_i) = \inf[\rho(x,y) \ ; \ y \in F_i]$. Suppose that $F_2 \subset \{x : \rho(x,F_1) \geq \varepsilon\}$ for some $\varepsilon > 0$. The function $x \to \beta(x,F_1,F_2) = \dfrac{\rho(x,F_2)}{\rho(x,F_1) + \rho(x,F_2)}$ takes values zero or F_2 and unity on F_1.

It is easily checked that it staisfies a Lipschitz condition with respect to ρ. More precisely $\beta \in \Lambda_\rho[1,4\varepsilon^{-1}]$. These functions will be used repeatedly below. We shall also need the following lemmas.

Lemma 4. Let f a real valued function defined on a subset A of $\mathcal{X}$. Suppose that $a \leq f \leq b$ and that $|f(x_1) - f(x_2)| < k\cdot\rho(x_1,x_2)$ for $x_i \in A$. Then f possesses an extension to $\mathcal{X}$ which satisfies the same conditions.

Proof Cipzer and Geher show in [8] that $g(x) = \inf\{f(y) + k\,\rho(x,y) \ ; \ y \in A\}$ gives an extension satisfying the Lipschitz condition. To obtain the same bounds take $(g \vee a) \wedge b$.

Lemma 5. Let γ be a real function defined on $\mathcal{X}$. Assume that $0 \leq \gamma \leq 1$ and that there is an $\varepsilon > 0$ and an integer m such that $\rho(x,y) < \varepsilon$ implies $|\gamma(x) - \gamma(y)| < \frac{1}{2m}$. Then there is an $f \in \Lambda_\rho[1,\frac{4}{\varepsilon}]$ such that $0 \leq f \leq \gamma \leq f + \frac{1}{m}$.

Proof For $j = 0,1,2,\cdots,m$ let A_j be the set $A_j = \{x ; \gamma(x) > \frac{j}{m}\}$. Let u_j be the function $u_j(x) = \beta[x,A_{j+1},A_j^c]$ which is unity on A_{j+1} and zero on A_j^c. Consider the function $f = \frac{1}{m} \sum_{j=1}^{m} u_j$. It is easily verified that $0 \leq f \leq \gamma \leq f + \frac{1}{m}$. Furthermore the sets A_j^c and A_{j+1} are at distance at least ε. Thus f is an element of $\Lambda_\rho[1,\frac{4}{\varepsilon}]$ as average of functions u_j which have this property.

Proposition 3 If the structure $\mathcal{U}$ on $\mathcal{X}$ is metrizable than the structure $[U]$ is metrizable on the norm bounded subsets of Λ^*.

Proof Suppose that ρ is a bounded metric inducing the structure $\mathcal{U}$ on $\mathcal{X}$. Let $||\mu||_\rho$ be the corresponding Dudley norm on Λ^*. It is clear that the structure induced by this norm is weaker than $[U]$. To show that they coincide on bounded sets let S be an U E B of Λ. Assume for simplicity that $\gamma \in S$ implies $0 \leq \gamma \leq 1$. For each $\gamma \in S$ construct an approximation $f_\gamma \in \Lambda_\rho[1,\frac{4}{\varepsilon}]$ according to lemma 5 so that $0 \leq f_\gamma \leq f_\gamma + \frac{1}{m}$. Since S is an U E B set the functions f_γ satisfy a Lipschitz condition $|f_\gamma(x) - f_\gamma(y)| \leq \frac{4}{\varepsilon} \rho(x,y)$ for some $\varepsilon > 0$ independent of γ. This gives

$|\langle\mu, f_\gamma\rangle - \langle\mu,\gamma\rangle| \leq \frac{1}{m} ||\mu||$ for every $\mu \in \Delta^*$ and every $\delta \in S$. The Lipschitz condition gives $|\langle\mu, f_\gamma\rangle| \leq \frac{4}{\varepsilon} ||\mu||_\rho$ and therefore

$$|\langle\mu,\gamma\rangle| \leq \frac{1}{m} ||\mu|| + \frac{4}{\varepsilon} ||\mu||_\rho$$

for every $\mu \in \Delta^*$ and $\gamma \in S$. Consider then a ball $D = \{\mu\ ;\ ||\mu|| \leq b/2\}$. For any given $\delta > 0$ one can select m such that $2b < m\delta$. This gives an ε occuring in the condition of lemma 5. If $\mu \in D-D$ and $||\mu||_\rho \leq \frac{1}{8}\varepsilon\delta$ we conclude that $|\langle\mu,\gamma\rangle| \leq \delta$ for all $\gamma \in S$. Hence the desired result.

Note 1 The structure [U] does not usually coincide with that induced by $||\mu||_\rho$ on $\mathcal{M}_u$ or on $\mathcal{M}_u^+$ since there usually exist uniformly continuous functions which do not satisfy a Lipschitz condition. Hence the topologies defined by [U] and $||\mu||_\rho$ or convex symmetric subsets of Δ^* or $\mathcal{M}_u$ may be different when these sets are unbounded.

Note 2 Assuming that $\{\mathcal{X},\mathcal{U}\}$ is metrizable, let $\Gamma = C^b(\mathcal{X})$ be the space of bounded continuous functions on . Let Γ^* be the dual of Γ for the uniform norm. One can extend any $\mu \in \Delta^*$ to the whole of Γ. In this situation the most common definition of convergence is the so called "weak convergence" for measures which is nothing but $W(\Gamma^*,\Gamma)$. Although for the positive cone $(\Gamma^*)^+$ of Γ^* the topology of $W(\Gamma^*,\Gamma)$ does not appear pathological, the uniform structure associated with it and consequently the topology of the unit ball $\{\mu \; ; \; \mu \in \Gamma^*, ||\mu|| \leq 1\}$ do not reflect in any reasonable way the uniform structure $\mathcal{U}$ as can be seen for instance when is the real line. In such a case [U] seems more usable. The following proposition is a variant of a well known theorem of Yu V. Prohorov [9] (see also [7] and [10] .

It should perhaps be noted that for every uniform space $(\mathcal{X}, \mathcal{U})$, the subsets of $\mathcal{M}_u$ which are [U] precompact are also $W[\mathcal{M}_u, \Delta]$ relatively compact and that, by proposition 2, the converse is true for subsets of $\mathcal{M}_u^+$. However no tightness condition similar to that of [9] can hold for in general for [U] precompact subsets since when $(\mathcal{X}, \mathcal{U})$ is the usual real line the sequence $\{\mu_n\}$ defined by $\langle\mu_n, \gamma\rangle = \gamma(n) - \gamma(n+\frac{1}{n})$ conferges to zero for [U] .

<u>Proposition 4</u> <u>Assume that</u> $(\mathcal{X}, \mathcal{U})$ <u>is metrizable</u>. <u>Let</u> B <u>denote the unit ball of</u> ▲ <u>with the topology of precompact convergence</u>. <u>A subset</u> A <u>of</u> $\mathcal{M}_p^+$ <u>is</u> [U] - <u>precompact if and only if the set of its restrictions to</u> B <u>is equicontinuous</u>.

<u>Remark</u> If A. is equicontinuous on B or precompact it is bounded for the norm. Also according to Ascoli's theorem a bounded set A is precompact if and only if its restrictions to U E B subsets of Δ (or equivalently, U E B subsets of B) are equicontinuous. The point of the proposition is that this

already implies equicontinuity on B itself at least when $(\mathcal{X},\mathcal{U})$ is metrizable and $A \subset \mathcal{M}_p^+$.

<u>Proof</u> Suppose A precompact for $[U]$ and $A \subset \mathcal{M}_p$. Take an $\varepsilon \in (0,1/2]$ and a metric ρ defining $\mathcal{U}$ or $\mathcal{X}$. We shall show that there exists a linear map T of $\mathcal{M}_p$ into itself such that

1) T and $I - T$ are positive

2) $\|(I - T)\mu\|_\rho < \varepsilon$ for all $\mu \in A$

3) $\{T\mu \; ; \; \mu \in A\}$ is equicontinuous on B. This implies the statement of proposition 4 since for <u>positive</u> μ we can write $\|(I - T)\mu\| = \|(I - T)\mu\|_\rho$.

For this purpose construct pairs of sets $\{K_n, G_n\}$ as follows. Take $K_o = \emptyset$, $G_o = \mathcal{X}$ and then for $n \geq 1$,

a) K_n precompact, $K_n \subset G_{n-1}$

b) $G_n = \{x : \rho(x,K_n) > \varepsilon^n\}$

Let u_n be the function equal to unity on K_n and zero on G_n^c defined by $u_n(x) = [\rho(x,G_n^c)][\rho(x,K_n) + \rho(x,G_n^c)]$. Let L_o be the identity map. For $n > 1$ let $L_n\mu$ be defined by $\langle L_n\mu, \gamma\rangle = \langle \mu, u_n\gamma\rangle$. Clearly $(I - L_n)$ and L_n are positive. The same is true of $T_n = L_n L_{n-1} \cdots L_1$ and $(I - T_n)$. Also T_n maps $\mathcal{M}_p$ into $\mathcal{M}_p$ and for every $\mu \in \mathcal{M}_p$ the sequence $T_n\mu$ converges in the norm of Δ^* to a limit $T\mu \in \mathcal{M}_p$. To prove this last statement consider only elements $\mu^+ \in \mathcal{M}_p^+$ and note that $T_n\mu^+ \leq T_{n-1}\mu^+$.

If T is constructed as described, let $K = \bigcup_n K_n$. This is a precompact set since for each n every element of K is within distance ε^n of the precompact set $\bigcup_{j\leq n} K_j$. If $\gamma \in B$ and $|\gamma(x)| \leq \delta/2$, $\delta > 0$ for all $x \in K$ then $|\gamma(x)| < \delta$ on some neighborhood V of K. However V must contain some G_m . Since u_m vanishes on G_m^c this gives $|<(T_n\mu),\gamma> | \leq \delta \,||\mu||$ for all $n \geq u$. Hence $|<T\mu,\gamma>| \leq \delta \,||\mu||$

To select the pairs $\{K_n, G_n\}$ so that $(I-T)$ has the desired property,note that for each n, the set T_nA is a precompact subset of $_p$. Hence there is a finite set $\{\mu_{n\cdot j}\ ;\ j = 1,2,\cdots,k_n\}$ such that $\sup_{\mu\in T_nA} \inf_j ||\mu-\mu_{n,j}||_\rho < \frac{\varepsilon^{2n+2}}{5}$. For these $\mu_{n\cdot j}$ we can find a precompact set K_{n+1} such that $\gamma \in B$, $\gamma(x) = 0$ for $x \in K_{n+1}$ implies $|<\mu_{n\cdot j},\gamma>| < \varepsilon^{n+1}$.

Define u_{n+1} as prescribed and let L_{n+1} be the multiplication by u_{n+1} . Then $||\mu_{n\cdot j} - L_{n+1}\,\mu_{n,j}|| < \varepsilon^{n+1}$ for all j . Also, $u_{n+1} \in \Lambda\,\rho[1,4\,\varepsilon^{-(n+1)}]$ for every $\mu \in T_n A$. By addition it follows that $||(I-T)\mu||_\rho \leq 2\,\varepsilon(1-\varepsilon)^{-1}$ for all $\mu \in A$.

It remains to check that the precompact set K_{n+1} can be taken contained in G_n . This is a consequence (for instance) of the fact that $T_n A$ can be considered a precompact subset of the space $\mathcal{M}_p(G_n)$ built on G_n instead of $\mathcal{X}$. The result follows.

<u>Corollary 1</u> <u>If</u> $\{\mathcal{X},\mathcal{U}\}$ <u>is metrizable then</u> $\mathcal{M}_p = \mathcal{M}_u$.

<u>Proof</u> According to lemma 2 it is sufficient to consider the positive elements of $\mathcal{M}_u$. Suppose then $\mu \in \mathcal{M}_u^+$. By proposition 3 and proposition 2 there is a sequence $\{\mu_n\}$ of element of $\mathcal{M}_s^+ \subset \mathcal{M}_p^+$ which converges to μ for $[U]$. Since a Cauchy sequence is precompact, proposition 4 implies that $\{\mu_n\}$ is equicontinuous on B. Hence $\mu \in \mathcal{M}_p^+$.

<u>Corollary 2</u> <u>Let</u> $\{\mathcal{X},\mathcal{U}\}$ <u>be an arbitrary uniform space</u>. <u>An element</u> $\mu \in \Delta^*$ <u>belongs to</u> $\mathcal{M}_u$ <u>if and only if for every metrizable space</u> $\{\mathcal{X}_1,\mathcal{U}_1\}$ <u>and every uniformly continuous map</u> φ <u>of</u> $\{\mathcal{X},\mathcal{U}\}$ <u>into</u> $\{\mathcal{X}_1,\mathcal{U}_1\}$ <u>the image</u> $\varphi\mu$ <u>of</u> μ <u>belongs to</u> $\mathcal{M}_p[\mathcal{X}_1,\mathcal{U}_1] = \mathcal{M}_u[\mathcal{X}_1,\mathcal{U}_1]$.

<u>Proof</u> That $\varphi\mu$ belongs to $\mathcal{M}_p[\mathcal{X}_1,\mathcal{U}_1]$ is clear. Conversely suppose $\mu \in \Delta^*$, $\mu \geq 0$. If S is an U E B subset of Δ, let ρ_s be the pseudo-metric it defines and let $\{\mathcal{X}_1,\mathcal{U}_1\}$ be the corresponding metric space. Let φ be the canonical map of $\mathcal{X}$ into $\mathcal{X}_1$. If $\varphi\mu \in \mathcal{M}_u[\mathcal{X}_1,\mathcal{U}_1]$ it is the limit uniformly on the U E B subsets of $\Delta[\mathcal{X}_1,\mathcal{U}_1]$ of elements of $\mathcal{M}_s^+[\mathcal{X}_1,\mathcal{U}_1]$. Since the transform of S is a subset of the Lipschitz functions on $\{\mathcal{X}_1,\mathcal{U}_1\}$ this shows that μ is limit

uniformly on S of elements of $\mathcal{M}_s^+$. The result follows.

Note According to the preceding proposition if $\mu \in \mathcal{M}_u$ then its uniformly continuous images on complete metric spaces extend to Radon measures. In particular they are σ-additive (σ-smooth). However the same need not be true of μ itself or if its images on metric spaces which are not complete. This can be seen for instance by taking for $\mathcal{X}$ the rationals of $[0,1]$ with the ordinary uniform structure. Then every $\mu \in \Delta^*$ belongs to $\mathcal{M}_p$ since $\mathcal{X}$ is precompact. The Lebesgue integral is an example of element of $(\Delta^*)^+$ which is not σ-smooth on Δ^*.

4. Characterizations by images into $C(K)$ spaces. According to corollary 2 of proposition 4 of the preceding section an element $\mu \in \Delta^*$ belongs to $\mathcal{M}_u$ if and only if its images by uniformly continuous maps into complete metric spaces are extendable to Radon measures. Such a characterization is not very convenient if $\mathcal{X}$ carries in addition to $\mathcal{U}$ other structures such as a group or vector space structure. For instance in the latter case one would like to use only linear maps into Banach or Frechet spaces. The present section will show that this is indeed sufficient. The results and the method of proof are not essentially different from those suggested in [2].

Definition Let $\{S_\alpha\}$ be a family of U E B subsets of Δ. This family will be called rich if on the set $\mathcal{X}$ itself the structure $\mathcal{U}$ is identical to the structure of uniform convergence on the sets S_α. ¶ Note that for any compact U E B set S the evaluation map gives a canonical map $x \to \tilde{x}$ of $\mathcal{X}$ into $C(S)$. Similarly if $\{S_\alpha\}$ is an arbitrary family of U E B sets one can form a locally compact set T topological sum of disjoint copies of the sets $\{S_\alpha\}$. Let then $C(T)$ be space of continuous real functions defined on T with the topology of uniform convergence on the compacts of T. The evaluation map gives again a canonical map of $\mathcal{X}$ into $C(T)$. If the number of sets in the family $\{S_\alpha\}$ is countable this space $C(T)$ is a Frechet space. In all these cases if $\mathcal{K} = \{S_\alpha\}$ is any family of compact U E B subsets of Δ we shall say that the corresponding Banach, Frechet or $C(T)$ spaces are associated with $\mathcal{K}$.

Lemma 6 Let K be a compact space and let f be a uniformly continuous map of $\mathcal{X}$ into the Banach space $C(K)$ of continuous functions on K. Let $\mathcal{K}$ be a rich family of compact U E B subsets of Δ. There is a countable subfamily $\{S_n\} \subset \mathcal{K}$ such that if $C(T)$ is the Frechet space constructed from the sequence $\{S_n\}$ then the map f factorizes in the form $f = g \circ h$ where h is a uniformly continuous map of $\mathcal{X}$ into $C(T)$ and g is a uniformly continuous map of $C(T)$ into $C(K)$.

<u>Proof</u> For each integer n there is a finite set $\{S_{n,j}\}$ with $S_{n,j} \in \mathcal{X}$ and an integer k such that $\sup_j \sup\{|\gamma(x)-\gamma(y)|$; $\gamma \in S_{n,j}\} < 1/k$ implies $|f(x)-f(y)| < 1/n$. Take for T the topological sum of disjoint copies of the $S_{n,j}$. Let $x \to h(x)$ be the canonical map of $\mathcal{X}$ into $C(T)$. If $h(x) = h(y)$ then $f(x) = f(y)$. Thus one can write $f = g \circ h$ for some map g of $h(\mathcal{X})$ into $C(K)$. By construction g is uniformly continuous. Hence it extends by continuity to the set $A = \overline{h(\mathcal{X})}$ closure of $h(\mathcal{X})$ in $C(T)$. Consider then the product $A \times K \subset C(T) \times K$. The map g can be identified to a uniformly continuous real function defined on $A \times K$. This function extends to a uniformly continuous function defined on the whole of $C(T) \times K$ either by application of a result of Katetov [11] or by observing that g is already uniformly continuous on a metric quotient of $A \times K$. This gives the desired result.

<u>Note</u> If f was a map into a metric space $\mathcal{Y}$ one could first imbed $\mathcal{Y}$ isometrically into a space $C(K)$ and then apply the preceding lemma. One can for instance take for K the space of Lipschitz functions $\gamma, |\gamma| < 1$ such that $|\gamma(y_1)-\gamma(y_2)| \leq$ dist(y_1,y_2).

<u>Lemma 7</u> <u>Let</u> μ <u>be an element of</u> Δ^* . <u>Let</u> $\mathcal{K}$ <u>be a rich increasingly directed family of compact</u> U E B <u>subsets of</u> Δ. <u>Then</u> μ <u>is already uniquely determined on</u> Δ <u>by its canonical</u>

images on the Banach spaces $C(K)$ for $K \in \mathcal{K}$.

Proof Let $f \in \Delta$. Write $f = g \circ h$ for a map h of $\mathcal{X}$ into a Ferchet space $C(T)$ of $\mathcal{K}$. The value $\langle \mu, f \rangle$ is well determined if the image $\nu = h(\mu)$ of μ on $C(T)$ is given.

Order the copies of the sets of $\mathcal{K}$ which form T in a sequence $\{S_n\}$. Let $T_n = \bigcup_{j \leq n} S_j$. For each n let Π_n be the projection of $C(T)$ into $C(T)$ given by $(\Pi_n z)(t) = z(t)$ if $t \in T_n$ and $(\Pi_n z)(t) = 0$ otherwise. If ξ is a uniformly continuous bounded real function defined on $C(T)$ then for each $\varepsilon > 0$ there is an n such that $|\xi(z) - \xi(\Pi_n z)| < \varepsilon$. Thus $|\langle \nu, \xi \rangle - \langle \nu, (\xi \circ \Pi_n) \rangle| \leq \varepsilon ||\nu||$. This gives $\langle \nu, \xi \rangle = \lim \langle \Pi_n \nu, \xi \rangle$ and proves the desired result since one can also consider Π_n as a map of $C(T)$ onto $C(T_n)$.

Lemma 8 Let $\mathcal{K}$ be a rich family of compact U E B subsets of Δ. For each Frechet space $C(T)$ associated to $\mathcal{K}$ consider the canonical map φ of $\mathcal{X}$ into $C(T)$ and the $[U]$ structure on $\Delta^*[C(T)]$. The structure $[U]$ on Δ^* is the smallest which makes the canonical maps of Δ^* into all the $\Delta^*[C(T)]$ uniformly continuous. Also in order that a filter $\mathcal{F}$ on Δ^* verges to $\mu \in \Delta^*$ for $W[\Delta^*, \Delta]$ (or $[U]$) it is necessary con- and sufficient that its canonical images in the $\Delta^*[C(T)]$ spaces of $\mathcal{K}$ converge correspondingly to the images of μ.

Proof This is an immediate consequence of lemma 6 since each U E B subset of Δ defines a uniformly continuous map into a metric space.

Lemma 9 Assume that $\mathcal{K}$ i... rich and increasingly directed. Let $\mathcal{F}$ be a filter on Δ^*. Assume that $\mathcal{F}$ contains either a bounded set or a set of positive elements of Δ^*. Then $\mathcal{F}$ convergesto μ for $W[\Delta^*,\Delta]$ if and only if for each Banach space of $\mathcal{K}$ the image of $\mathcal{F}$ converges correspondingly to the image of μ .

Proof Since bounded subsets of Δ^* are $W[\Delta^*,\Delta]$ relatively compact that result follows immediately from lemma 7.

Proposition 5 Let $\mathcal{K}$ be an arbitrary rich family of compact U E B subsets of Δ. Let A be a subset of the positive case $\mathcal{M}_u^+$. Then A is [U] (or equivalently $W[\Delta^*,\Delta]$) relatively compact in $\mathcal{M}_u^+$ if and only if its canonical images on the Banach spaces $C(K)$, $K \in \mathcal{K}$ are tight.

Proof The necessity follows immediately from proposition 4. Conversely, note that according to lemma 6 and Ascoli's theorem it is sufficient to show that the images of A on the Frechet spaces of $\mathcal{K}$ are tight.

Suppose then that T is a direct sum $T = \bigcup S_n$ as before. For each n let π_n be the projection $(\pi_n z)(t) = z(t)$ for $t \in S_n$ and $(\pi_n z)(t) = 0$ for $t \in S_n^c$. Let A_1 be the image of A on $C(T)$. By assumption each $\pi_n A_1$ is tight. Thus for each n there is a convex symmetric compact $H_n \subset C(S_n)$ such that if $\gamma \in \Delta[C(S_n)]$ vanishes on H_n and $|\gamma| \leq 1$ then $|\langle\mu,\gamma\rangle| < \varepsilon^n(1-\varepsilon)$ for all $\mu \in \pi_n A_1$. Consider the cartesian product H of the sets H_n. This can be identified to a compact subset of $C(T)$. It is now easily verifiable that if $\gamma \in \Delta[C(T)]\pi$, $0 \leq \gamma \leq 1$, vanishes on H then $|\langle\mu,\gamma\rangle| \leq \varepsilon$ for each $\mu \in A_1$. (For this reduce the problem to a finite sum $T_n = \bigcup_{j \leq n} S_j$ as in lemma 7. For $1 \leq m \leq n$ let U_m be a function $U_m \in \Delta[C(T)]$ such that U_m vanishes on $\{z ; \pi_m z \in H_m\}$ and such that $0 \leq U_m \leq 1$ otherwise. Write $\gamma = \gamma U_n + \gamma(1-U_n) = \gamma U_n + \gamma(1-U_n) U_{n-1}$ and so forth.) The statement that $0 \leq \gamma \leq 1$, $\gamma(z) = 0$ for $z \in H$ implies $|\langle\mu,\gamma\rangle| < \varepsilon$ is already sufficient to imple the desired tightness. Hence the result.

<u>Note 1</u> The reader will note that lemma 8 gives another proof of proposition 3.

<u>Note 2</u> As an example of case in which proposition 5 is applicable consider two vector spaces $\mathcal{X}$ and $\mathcal{Y}$ in duality. Let $\mathcal{U}$ be the $W[\mathcal{X},\mathcal{Y}]$ structure of $\mathcal{X}$. Then $\mathcal{M}_u$ coincides with the so called "weak distributions" or $\mathcal{X}$. That is measures whose continuous linear images in Euclidean spaces are σ-additive. A subset A of $\mathcal{M}_u^+$ is relatively compact if and only if for each $y \in \mathcal{Y}$, the images of A by $x \to \langle y,x\rangle$ are relatively compact on the line.

5. <u>Some results relative to paracompact spaces.</u> The preceding

sections were conserned with a vector lattice Δ of bounded functions on a set. This Δ was subject only to the restrictions that $I \in \Delta$ and that Δ be complete for the uniform norm. In [10] Alexandroff used lattices Δ which can be characterized by the additional property that they are their own $U_\sigma \cap L_\sigma$ hull. One can also consider lattices Δ subject to the further restriction that they are their own $U_\tau \cap L_\tau$ hull. (Here as in [12] $U_\sigma = (-L_\sigma)$ is the set of functions which are pointwise suprema of countable subsets of Δ. The set $U_\tau = (-L_\tau)$ is obtained by removing the countability restriction). Given a lattice Δ its $U_\tau \cap L_\tau$ hull is exactly the set $C^b(\mathcal{X})$ of bounded functions which are continuous for the topology induced by Δ. Taking this into account we shall consider throughout this section the following situation.

<u>The space $\mathcal{X}$ is a completely regular topological space. The uniform structure $\mathcal{U}$ is the universal structure and Δ. is the space of bounded continuous functions on $\mathcal{X}$.</u>

Let us recall that the universal structure $\mathcal{U}$ is the one defined by all the continuous pseudo-metrics on $\mathcal{H}$. It is then clear that uniform equicontinuity on $(\mathcal{H}, \mathcal{U})$ is the same thing as continuity on the topological space $\mathcal{X}$. Thus all the statements in the preceding section can be automatically translated. Note however that the statements of section 4 are not to be replaced by statements involving families of equicontinuous sets which generate the <u>topology</u> of $\mathcal{X}$ but by statements involving families of equicontinuous sets which generate $\mathcal{U}$. The spaces $\mathcal{X}$ and Δ being as assumed here we shall make use of the following definition.

Definition Let $\{u_\alpha ; \alpha \in A\}$ be a family of elements $u_\alpha \in \Delta$ indexed by a set A. This is called a partition of unity (or more precisely a Δ-partition of unity)

1) $0 \leq u_\alpha \leq I$ for each $\alpha \in A$

2) $\sum_\alpha u_\alpha(x) = 1$ for each $x \in \mathcal{X}$.

Let $\{u_\alpha ; \alpha \in A\}$ be a partition of unity. Let M_A be the space of bounded real valued functions $\alpha \leadsto \beta(\alpha)$ defined on A. For each $\beta \in M_A$ let $T'\beta$ be the function defined on $\mathcal{H}$ by

$$(T'\beta)(x) = \sum_\alpha \beta(\alpha)\ U_\alpha(x) \ .$$

Lemma 10 If $\{u_\alpha ; \alpha \in A\}$ is a Δ-partition of unity the map $\beta \leadsto T'\beta$ is a linear map from M_A to Δ which transforms the unit ball $\{\beta ; \beta \in M_A , |\beta| \leq I\}$ of M_A into an equicontinuous subset of Δ. The transpose map T transforms the space $\mathcal{M}_u$ built on $\{\mathcal{X}, \mathcal{U}\}$ into the space L of bounded Radon measures on the discrete space A.

Proof Let x be an element of $\mathcal{X}$. For any $\varepsilon > 0$ there is a finite set $F \subset A$ such that $\sum_{\alpha \in F} u_\alpha(x) > 1 - \varepsilon/4$. The function $\leadsto g(y) = \sum_{\alpha \in F^c} u_\alpha(y) = 1 - \sum_{\alpha \in F} u_\alpha(y)$ is continuous on $\mathcal{X}$. Hence there is a neighborhood V of x such that $g(y) < \frac{\varepsilon}{2}$ for all $y \in V$. This implies $|\sum_{\alpha\in F^c}\beta(\alpha)u_\alpha(y)| < ||\beta|| \sum_{\alpha\in F^c} u_\alpha(y) < \varepsilon/2$ for all β in the unit ball B of M_A. The first result follows. For the second note that for fixed x the map $\beta \mapsto \sum_\alpha \beta(\alpha)u_\alpha(x)$ is a bounded Radon measure on A. If $\mu \in \mathcal{M}_u$ then μ is a limit uniformly on the U E B sets of Δ of measures μ_v what have finite support. The images $T\mu_v$ converge uniformly on $B = \{\beta ; \beta \in M_A , |\beta| \leq I\}$. Equivalently the measure $T\mu_v$ converge for the usual L_1-norm of L. The result follows.

<u>Corollary</u> <u>If</u> $\mu \in \mathcal{M}_u$ <u>and</u> $\gamma \in \Delta$ <u>then for every partition of unity</u> $\{ _\alpha ; \alpha \in A\}$ <u>the value</u> $\langle\mu,\gamma\rangle$ <u>is the sum</u> $\sum_\alpha \langle\mu,\gamma u_\alpha\rangle$ <u>limit of sums</u> $\sum_{\alpha\in F} \langle\mu,\gamma u_\alpha\rangle$ <u>along the filter of finite subsets of</u> A.

<u>Proof</u> One can assume that $\mu \geq 0$ and $\gamma \geq 0$. Define $\nu \in \mathcal{M}_u$ by $\langle\nu, f\rangle = \langle\mu,\gamma f\rangle$ for $f \in \Delta$. Consider the image $\varphi = T\nu$ of ν by the transpose of the map $\beta \mapsto \sum \beta(\alpha)\, u_\alpha$. Since φ is a Radon measure for each $\varepsilon > 0$ there is a finite set F such that $\varphi(F^c) < \varepsilon$ or equivalently $\sum_{\alpha\in F^c}\langle\varphi, u_\alpha\rangle < \varepsilon$. Since $||\varphi|| = \sum_{\alpha\in A} \langle\varphi, _\alpha\rangle$ the result follows.

<u>Lemma 11</u> <u>Let</u> $\mathcal{X}$ <u>be a paracompact space with its universal uniform structure</u> $\mathcal{U}$. <u>Let</u> $\Delta = C^b(\mathcal{X})$. <u>A bounded linear functional</u> φ <u>defined on</u> Δ <u>is</u> τ-smooth <u>on</u> Δ <u>if and only</u>

<u>if it possesses the following property</u> (π):

<u>For every</u> $\varepsilon > 0$, <u>every</u> $\gamma \in \Delta$, $0 \leq \gamma \leq I$ <u>and every locally finite partition of unity</u> $\{u_\alpha ; \alpha \in A\}$ <u>there is a finite set</u> $F \subset A$ <u>such that</u>

$$|\langle\varphi,\gamma(\sum_{\alpha\in F^c} u_\alpha)\rangle| < \varepsilon .$$

<u>Proof</u> The necesity of condition (π) is obvious. To prove the sufficiency suppose that $\varphi = \varphi_1 + \varphi_2$ with φ_1 element of the space $\mathcal{M}_\tau$ of τ-smooth functionals and φ_2 disjoint from $\mathcal{M}_\tau$. Suppose for instance that $0 \leq ||\varphi_2^-|| \leq ||\varphi_2^+||$. Let ε' be a number $\varepsilon' \in (0,1)$. Let $\gamma \in \Delta$, $0 \leq \gamma \leq I$ be such that $\langle\varphi_2^+,(1-\gamma)\rangle + \langle\varphi_2^-,\gamma\rangle < \varepsilon'$. Let $\hat{\mathcal{X}}$ be the Stone-Cech compactification of $\mathcal{X}$. Since φ_2 is disjoint from $\mathcal{M}_\tau$, there is a ψ such that 1) $0 \leq \psi \leq \varphi_2^+$ and 2) $(1-\varepsilon')||\varphi_2^+|| \leq ||\psi||$ and 3) ψ has a compact support $K \subset \hat{\mathcal{X}} \setminus \mathcal{X}$. For each $x \in \mathcal{X}$ let V_x be an open neighborhood of x in $\hat{\mathcal{X}}$ such that the closure $\overline{V}_x$ be disjoint from K. The family $\{V_x \cap \mathcal{X} ; x \in \mathcal{X}\}$ is an open cover of $\mathcal{X}$. Since $\mathcal{X}$ is para-compact there is a locally finite partition of unity $\{v_x ; x \in \mathcal{X}\}$ such that $v_x(y) = 0$ if $y \notin V_x \cap \mathcal{X}$. It follows that $\langle\psi,v_x\rangle = 0$ for every $x \in \mathcal{X}$. Suppose now that condition (π) is satisfied. Take F so large that $\langle|\varphi_1|,\gamma(\sum_{x\in F^c} v_x)\rangle < \varepsilon'$ and $\langle\varphi,\gamma \sum_{x\in F^c} v_x)\rangle < \varepsilon$. Writing $g = \sum_{x\in F^c} v_x$ this gives

$$\langle\varphi_2^+,\gamma g\rangle < \varepsilon + \langle\varphi_2^-,\gamma\rangle + \varepsilon' < \varepsilon + 2\,\varepsilon'$$

thus $\langle\psi,\gamma g\rangle < \varepsilon + 2\,\varepsilon'$. However since $\langle\psi,v_x\rangle = 0$ for each x , one can replace g by $I = \sum_{x\in F} v_x + g$ in the last inequality. Thus $\langle\psi,\gamma\rangle = ||\psi|| - \langle\psi,1-\gamma\rangle < \varepsilon + 2\,\varepsilon'$ and finally $||\psi|| < \varepsilon + 3\,\varepsilon'$. Since ε' is arbitrary we can conclude that $||\varphi_2|| < 2\,\varepsilon$. This completes the proof of the lemma.

<u>Corollary</u> <u>If $\mathcal{X}$ is paracompact the spaces $\mathcal{M}_u$ and $\mathcal{M}_\tau$ coincide.</u>

This is an immediate consequence of lemmas (10) and (11) above.

<u>Proposition 6</u> <u>Let $\mathcal{X}$ be an arbitrary completely regular space with universal uniform structure $\mathcal{U}$. Let $\Delta = C^b(\mathcal{X})$ and let φ be a bounded linear functional on Δ. The following conditions are all equivalent.</u>

1) $\varphi \in \mathcal{M}_u$

2) <u>for every partition of unity $\{u_\alpha\ ;\ \alpha \in A\}$ every $\gamma \in \Delta$, $0 \leq \gamma \leq I$ and every $\varepsilon > 0$ there is a finite set $F \subset A$ such that</u> $|\langle\varphi,\gamma(\sum_{\alpha\in F^c} u_\alpha)\rangle| < \varepsilon$.

3) <u>Same as (2) but for locally finite partitions of unity.</u>

4) <u>If f is a continuous map from $\mathcal{X}$ into a paracompact space $\mathcal{Y}$ then the image of $f\varphi$ of φ by</u>

f <u>is</u> τ-smooth <u>on</u> $C^b(\mathcal{Y})$.

5) <u>Same as</u> 4 <u>but with</u> $\mathcal{Y}$ <u>metric instead of para-compact</u>.

<u>Proof</u> According to lemmas 10 and 11 condition 1 implies all the other ones. Also $2 \Rightarrow 3 \Rightarrow (4) \Rightarrow (5)$ since for every locally finite partition of unity $\{u_\alpha, \alpha \in A\}$ on $\mathcal{Y}$ the functions $\{u_\alpha \circ f ; \alpha \in A\}$ form a locally finite partition of unity on $\mathcal{X}$. Thus it is sufficient to show that $(5) \Rightarrow (1)$. However this is for instance a consequence of corollary 2 of proposition 4 and lemma 11.

<u>Corollary</u> <u>Let</u> $\mathcal{X}$ <u>be a completely regular space with universal structure</u> $\mathcal{U}$. <u>Let</u> $\Delta = C^b(\mathcal{X})$, <u>and let</u> S <u>be a norm bounded subset of</u> $\mathcal{M}_u$. <u>Then the following conditions are all equivalent.</u>

1) S <u>is</u> $W[\mathcal{M}_u, \Delta]$ <u>countably relatively compact in</u> $\mathcal{M}_u$.

2) S <u>is</u> $W[\mathcal{M}_u, \Delta]$ <u>relatively compact in</u> $\mathcal{M}_u$

3) <u>For each locally finite partition of unity</u> $\{u_\alpha ; \alpha \in A\}$ <u>each</u> $\gamma \in \Delta$, $0 \leq \gamma \leq I$ <u>and each</u> $\varepsilon > 0$ <u>there is a finite set</u> F <u>such that</u>

$$\sup_{\mu \in S} |\langle \mu, (\sum_{\alpha \in F^c} u_\alpha \gamma) \rangle| < \varepsilon$$

4) The partition $\{u_\alpha\}$ and γ being as in (3) for each ε there is a finite set F such that

$$\sup_{\mu \in S} \sum_{\alpha \in F^c} |\langle \mu, \gamma u_\alpha \rangle| < \varepsilon .$$

Furthermore, in (3) and (4) the qualifications "locally finite" can be omitted.

Proof The equivalence of (1) and (2) has already been encountered more generally in corollary 1 of proposition 1. It is also clear from the corollary of lemma 10 that (4) implies (3). Assume then that (3) is satisfied and let $\mathcal{F}$ be a filter on S which converges for $W[\Delta^*,\Delta]$ to some $\lambda \in \Delta^*$. Let F be the set of condition (3). Then, along $\mathcal{F}$ we have $\lim \langle \mu, \sum_{\alpha \in F^c} \gamma u_\alpha) \rangle = \langle \lambda, (\sum_{\alpha \in F^c} \gamma u_\alpha) \rangle$, hence $|\langle \lambda, (\sum_{\alpha \in F^c} \gamma u_\alpha) \rangle| < \varepsilon$. It follows that $\lambda \in \mathcal{M}_u$ by proposition 6. To complete the proof it will be sufficient to show that (1) implies (3). Let T be the transpose of $\beta \mapsto \Sigma\, \beta(\alpha)\, u_\alpha$ as usual. Then T S is relatively countably compact in the space L of Radon measures on A. It follows then from a special case of a result of Grothendieck [13] (or can easily been verified directly be application of the Eberlein-Smulian theorems and the Baire category theorem) that there is a finite set F A such that $|v| (F^c) < \varepsilon$ for all $v \in T\,S$. This implies (4) and therefore (3).

Note According to the proof of this corollary condition (3) can also be replaced by apparently stronger condition that for each $\varepsilon > 0$ there is a finite set F such that

$$\sup_{\mu \in S} |\langle \mu, (\sum_{\alpha \in G^c} u_\alpha \gamma) \rangle| < \varepsilon$$

for any finite set G such that $F \subset G$.

<u>Proposition 7</u> <u>Let</u> $\mathcal{Y}$ <u>be completely regular with universal structure</u> $\mathcal{U}$. <u>Let</u> $\Delta = C^b(\mathcal{Y})$. <u>Let</u> S <u>be a</u> $W[\mathcal{M}_u, \Delta]$ <u>relatively compact subset of</u> $\mathcal{M}_u$. <u>Then the set</u> $S^+ = \{\mu^+; \mu \in S\}$ <u>is also relatively compact in</u> $\mathcal{M}_u$.

<u>Proof</u> One can assume $\sup\{||\mu|| \; ; \mu \in S\} \leq 1$. Also if S^+ is not relatively compact there is already a countable subset S_o of S such that S_o^+ is not relatively compact. Thus we can assume that S itself is countable. Let $\{u_\alpha \; ; \alpha \in A\}$ be a locally finite partition of unity such that for some $\eta > 0$ $\sup_{\nu \in S} \langle \nu^+, \sum_{\alpha \in F^c} \sigma \rangle > \eta$ for every finite set $F \subset A$.

For any given integer k and each $\nu \in S$ there is a finite set $A_{\nu,k}$ such that $\langle |\nu| \; ; \sum_{\alpha \in A^c_{\nu,k}} u_\alpha \rangle < 2^{-(k+2)}$.. Also there is an $f_{\nu,k} \in \Delta$, $0 \leq f_{\nu,k} \leq I$ such that $\langle \nu^+, (1-f_{\nu,k}) \rangle + \langle \nu^-, f_{\nu,k} \rangle < 2^{-(k+2)}$. Let $g_{\nu,k} = f_\nu \sum_{\alpha \in A_{\nu,k}} u_\alpha$ and $h_{\nu,k} = (1-f_{\nu,k}) \sum_{\alpha \in A_{\nu,k}} u_\alpha$. Define $g_{\nu,k} \cdot \nu$ by $\langle g_{\nu,k} \cdot \nu, \gamma \rangle = \langle \nu, g_{\nu,k} \gamma \rangle$.

A simple computation shows that $||g_{\nu,k} \cdot \nu - \nu^+|| \leq 2^{-k}$ and similarly $||h_{\nu,k} \cdot \nu + \nu^-|| \leq 2^{-k}$.

Let $D = \bigcup \{A_{\nu,k} ; \nu \in S, k=1,2\cdots\}$ and let $w = \sum_{\alpha \in D} u_\alpha$.

Let H_o be the smallest algebra which contains the constants, the function w, the u_α , $\alpha \in D$, .all the functions $f_{\nu,k}$ $g_{\nu,k}$ and $h_{\nu,k}$ and is closed for the uniform norm. This is a separable subalgebra of Δ . The smallest uniform structure which makes the functions of H_o uniformly continuous can be induced by one pseudo-metric ρ . Let $\mathcal{Y}$ be the metric space quotient of $\mathcal{X}$ for this pseudo-metric space quotient of for this pseudo-metric. Then $\mathcal{Y}$ is a separable metric space. Let H be the space $C^b(\mathcal{Y})$ of bounded ρ continuous functions on $\mathcal{Y}$. One can identify H_o to a subalgebra of H. Let ξ be the canonical map from $\mathcal{X}$ to $\mathcal{Y}$. Since ξ is continuous the image of S by ξ is countably relatively compact for $W[\mathcal{M}_\tau, C^b(\mathcal{Y})]$ in the space $\mathcal{M}_\tau$ on $\mathcal{Y}$. Consider the elements of S as linear functionals on H_o. Let ν^* be the positive part of ν on H_o. That is ν^* is defined by $\langle\nu^*,\gamma\rangle = \sup\{\langle\nu,f\rangle ; 0 \leq f \leq \gamma, f \in H_o\}$ for each $\gamma \in H_o$. By construction $\langle\nu^+ - \nu^*, \gamma\rangle < 2^{-k}$ for $0 \leq \gamma \leq I$. Therefore the set S is also such that $\sup_{\nu \in S} \langle\nu^*, \sum_{\alpha \in F^c} u_\alpha\rangle > \eta$ whatever may be the finite set F A. One can identify each u_α , $\alpha \in D$ to an element of $C^b(\mathcal{Y})$, say u'_α and then the functions $(1-w')$, $\{u'_\alpha , \alpha \in D\}$ form a partition of unity on $\mathcal{Y}$.

This reduces the problem to the case where the underlying space $\mathcal{X}$ is separable metric. For this case the result appears is well-known, see thm 28 of [14].

<u>Corollary 1</u> <u>A subset of</u> $\mathcal{M}_u$ <u>is</u> $W(\mathcal{M}_u,\Delta)$ <u>relatively compact in</u> $\mathcal{M}_u$ <u>if and only if it is relatively compact for the topology of uniform conveyence on the bounded equicontinuous subsets of</u> Δ.

To state a further corollary note that if $\mu \geq 0$ is τ-smooth on a space $C^b(\mathcal{X})$ one can extend it to lower semi-continuous functions by the procedure of Mac-Shane. We shall assume this done in the next statement.

<u>Proposition 8</u> <u>Let</u> $\mathcal{X}$ <u>be a paracompact space with its space</u> $\Delta = C^b(\mathcal{X})$. <u>For any norm bounded subset</u> $S \subset \mathcal{M}_\tau$ <u>the following conditions are all equivalent.</u>

1) S <u>is countably relatively</u> $W(\mathcal{M}_\tau,\Delta)$ <u>compact on</u> $\mathcal{M}_\tau$

2) S <u>is relatively</u> $W(\mathcal{M}_\tau,\Delta)$ <u>compact in</u> $\mathcal{M}_\tau$

3) <u>The set</u> $\{|\mu| \; ; \; \mu \in S\}$ <u>is relatively</u> $W(\mathcal{M}_\tau,\Delta)$ <u>compact in</u> $\mathcal{M}_\tau$

4) <u>For every decreasingly directed family</u> $\{f_v \; ; \; f_v \in C^b(\mathcal{X})\}$ <u>which decreases to zero pointwise on</u> $\mathcal{X}$ <u>one has</u>

$$\lim_{v} \sup_{\mu\in S} |\langle\mu, f_v\rangle| = 0$$

5) <u>For every open cover</u> $\{G_\alpha \; ; \; \alpha \in A\}$ <u>of</u> $\mathcal{X}$ <u>and every</u> $\varepsilon > 0$ <u>there is a finite subset</u> $F \subset A$ <u>such</u>

that

$$||\mu|| - |\mu| \bigcup [G_\alpha \; ; \; \alpha \in F] < \varepsilon$$

for all $\mu \in S$.

Proof We already know that $(1) \Leftrightarrow (2) \Rightarrow (3)$. To show that $(2) \Rightarrow (4)$ suppose first that S consist only of positive elements of $\mathcal{M}_\tau$ and is compact. Then the functions $\mu \mapsto \langle \mu, f_\nu \rangle$ decrease pointwise to zero on the compact S. Thus they converge uniformly according to Dini's theorem. Thus $(3) \Rightarrow (4)$ with $|\mu|$ instead of μ. This is stronger then(4). However (4) without the absolute value sign already implies (2) since for any locally finite partition of unity $\{u_\alpha \; ; \; \alpha \in A\}$ and any $\gamma \in \Delta$, $0 \leq \gamma \leq I$, the finite sums $(\gamma - \gamma \sum_{\alpha \in F} u_\alpha)$ decrease pointwise to zero. Thus $(4) \Rightarrow (2) \Rightarrow 3 \Rightarrow (4)$. For condition 5 note that there is a partition of unity $\{u_\alpha \; ; \; \alpha \in A\}$ subordinated to the cover G_α. Thus $(2) \Rightarrow (5) \Rightarrow (4)$. Hence the result.

Corollary <u>If $\mathcal{X}$ is paracompact and if S is relatively compact in $\mathcal{M}_\tau$, there is a Lindelöf subspace of $\mathcal{X}$, say $\mathcal{Y}$ such that $|\mu|(\mathcal{Y}^c) = 0$ for all $\mu \in S$.</u>

Proof One can assume that $\mu \geq 0$ for all $\mu \in S$. Let F be the intersection of the closed subsets F' such that $\mu(F') = ||\mu||$ for $\mu \in S$. Since $\mu \in \mathcal{M}_\tau$, one has also $\mu(F) = ||\mu||$ for $\mu \in S$. Let $\{G_\alpha \; ; \; \alpha \in A\}$ be an open cover of F. There is a partition of unity formed by a function u_o and functions $\{u_\alpha ; \alpha \in A\}$ such that u_o has support in F^c and such that each u_α has support in G_α. Noting that $\langle\mu, u_o\rangle = 0$, one can find for each integer k a finite subset A_k of A such that $\{\sum_\alpha \langle\mu, u_\alpha\rangle ; \alpha \prec A_k^c\} < 2^{-k}$ for $\mu \in S$. Let $A' = \bigcup_k A_k$ and let $v = \{\Sigma u_\alpha ; \alpha \in A'\}$. By construction $\langle\mu, 1-v\rangle = 0$. Suppose that $x \in F$ is such that $v(x) < 1$. Then there is an $\varepsilon > 0$ and a neighborhood V of x such that $v(y) < 1 - \varepsilon$ for all $y \in V$. By assumption $\mu(V) > 0$ for some $\mu \in S$. This would give $\langle\mu, 1-v\rangle > 0$ contary to the previous equality. It follows that $v(x) = 1$ for $x \in F$ and that $\{\bigcup G_\alpha ; \alpha \in A\} \supset F$. The conclusion follows.

It should perhaps be mentioned that there are spaces $\mathcal{X}$ such that $\mathcal{M}_\tau$ is strictly contained in $\mathcal{M}_u$. An example is provided by the space of countable ordinals. The measure which would correspond to a mass unity placed on the first uncountable ordinal is in $\mathcal{M}_u$ but not in $\mathcal{M}_\tau$. Of course proposition 7 does not apply to $\mathcal{M}_\tau$ on this particular space but only to $\mathcal{M}_u$.

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