14. Suppose that under \( H_0 \), a measurement \( X \) is \( N(0, \sigma^2) \), and that under \( H_1 \), \( X \) is \( N(1, \sigma^2) \) and that the prior probability \( P(H_0) = 2 \times P(H_1) \). As in Section 9.1, the hypothesis \( H_0 \) will be chosen if \( P(H_0|x) > P(H_1|x) \). For \( \sigma^2 = 0.1, 0.5, 1.0, 5.0 \):
   a. For what values of \( X \) will \( H_0 \) be chosen?
   b. In the long run, what proportion of the time will \( H_0 \) be chosen if \( H_0 \) is true? What is the probability of \( H_0 \) being chosen if \( H_0 \) is not true?

15. Suppose that under \( H_0 \), a measurement \( X \) is \( N(0, \sigma^2) \), and that under \( H_1 \), \( X \) is \( N(1, \sigma^2) \) and that the prior probability \( P(H_0) = P(H_1) \). For \( \sigma = 1 \) and \( x \in [0, 3] \), plot and compare (1) the \( p \)-value for the test of \( H_0 \) and (2) \( P(H_0|x) \). Can the \( p \)-value be interpreted as the probability that \( H_0 \) is true? Choose another value of \( \sigma \) and repeat.

16. In the previous problem, with \( \sigma = 1 \), what is the probability that the \( p \)-value is less than 0.05 if \( H_0 \) is true? What is the probability if \( H_1 \) is true?

17. Let \( X \sim N(0, \sigma^2) \), and consider testing \( H_0 : \sigma = \sigma_0 \) versus \( H_A : \sigma = \sigma_1 \), where \( \sigma_1 > \sigma_0 \). The values \( \sigma_0 \) and \( \sigma_1 \) are fixed.
   a. What is the likelihood ratio as a function of \( x \)? What values favor \( H_0 \)? What is the rejection region of a level \( \alpha \) test?
   b. For a sample, \( X_1, X_2, \ldots, X_n \) distributed as above, repeat the previous question.
   c. Is the test in the previous question uniformly most powerful for testing \( H_0 : \sigma = \sigma_0 \) versus \( H_1 : \sigma > \sigma_0 \)?

18. Let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables from a double exponential distribution with density \( f(x) = \frac{1}{\lambda} \exp(-\lambda|x|) \). Derive a likelihood ratio test of the hypothesis \( H_0 : \lambda = \lambda_0 \) versus \( H_1 : \lambda = \lambda_1 \), where \( \lambda_0 \) and \( \lambda_1 > \lambda_0 \) are specified numbers. Is the test uniformly most powerful against the alternative \( H_1 : \lambda > \lambda_0 \)?

19. Under \( H_0 \), a random variable has the cumulative distribution function \( F_0(x) = x^3, 0 \leq x \leq 1 \); and under \( H_1 \), it has the cumulative distribution function \( F_1(x) = x^2, 0 \leq x \leq 1 \).
   a. If the two hypotheses have equal prior probability, for what values of \( x \) is the posterior probability of \( H_0 \) greater than that of \( H_1 \)?
   b. What is the form of the likelihood ratio test of \( H_0 \) versus \( H_1 \)?
   c. What is the rejection region of a level \( \alpha \) test?
   d. What is the power of the test?

20. Consider two probability density functions on \([0, 1]\): \( f_0(x) = 1 \), and \( f_1(x) = 2x \). Among all tests of the null hypothesis \( H_0 : X \sim f_0(x) \) versus the alternative \( X \sim f_1(x) \), with significance level \( \alpha = 0.10 \), how large can the power possibly be?

21. Suppose that a single observation \( X \) is taken from a uniform density on \([0, \theta]\), and consider testing \( H_0 : \theta = 1 \) versus \( H_1 : \theta = 2 \).