EXAMPLE B  Let us calculate how much better optimal allocation is than proportional allocation for the population of hospitals. From Theorem C and Corollary A, we have

\[ \text{Var}(\bar{X}_{sp}) = \text{Var}(\bar{X}_{so}) + \frac{1}{n} \sum W_l (\sigma_l - \bar{\sigma})^2 \]

Therefore,

\[ \frac{\text{Var}(\bar{X}_{sp})}{\text{Var}(\bar{X}_{so})} = 1 + \frac{1}{n} \sum W_l (\sigma_l - \bar{\sigma})^2 \]

\[ = 1 + \frac{\sum W_l (\sigma_l - \bar{\sigma})^2}{(\sum W_l \sigma_l)^2} \]

Thus, under proportional allocation, the variance of the mean is about 20% larger than it is under optimal allocation.

We can also compare the variance under simple random sampling with the variance under proportional allocation. The variance under simple random sampling is, neglecting the finite population correction,

\[ \text{Var}(\bar{X}) = \frac{\sigma^2}{n} \]

In order to compare this equation with that for the variance under proportional allocation, we need a relationship between the overall population variance, \( \sigma^2 \), and the strata variances, \( \sigma_l^2 \). The overall population variance may be expressed as

\[ \sigma^2 = \frac{1}{N} \sum_{l=1}^L \sum_{i=1}^{N_l} (x_{il} - \mu)^2 \]

Also,

\[ (x_{il} - \mu)^2 = [(x_{il} - \mu) + (\mu_l - \mu)]^2 \]

\[ = (x_{il} - \mu)^2 + 2(x_{il} - \mu_l)(\mu_l - \mu) + (\mu_l - \mu)^2 \]

When both sides of this last equation are summed over \( l \), the middle term on the right-hand side becomes zero since \( N_l \mu_l = \sum_{i=1}^{N_l} x_{il} \), so we have

\[ \sum_{i=1}^{N_l} (x_{il} - \mu)^2 = \sum_{i=1}^{n_l} (x_{il} - \mu_l)^2 + N_l(\mu_l - \mu)^2 \]

\[ = N_l \sigma_l^2 + N_l(\mu_l - \mu)^2 \]

Dividing both sides by \( N \) and summing over \( l \), we have

\[ \sigma^2 = \sum_{l=1}^L W_l \sigma_l^2 + \sum_{l=1}^L W_l(\mu_l - \mu)^2 \]