Chapter 4 Expected Values

**THEOREM A**

Suppose that \( U = a + \sum_{i=1}^{n} b_i X_i \) and \( V = c + \sum_{j=1}^{m} d_j Y_j \). Then

\[
\text{Cov}(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_i d_j \text{Cov}(X_i, Y_j).
\]

This theorem has many applications. In particular, since \( \text{Var}(X) = \text{Cov}(X, X) \),

\[
\text{Var}(X + Y) = \text{Cov}(X + Y, X + Y)
\]

\[
= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)
\]

More generally, we have the following result for the variance of a linear combination of random variables.

**COROLLARY A**

\[
\text{Var}(a + \sum_{i=1}^{n} b_i X_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \text{Cov}(X_i, X_j).
\]

If the \( X_i \) are independent, then \( \text{Cov}(X_i, X_j) = 0 \) for \( i \neq j \), and we have another corollary.

**COROLLARY B**

\[
\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i), \text{ if the } X_i \text{ are independent.}
\]

Corollary B is very useful. Note that \( E(\sum X_i) = \sum E(X_i) \) whether or not the \( X_i \) are independent, but it is generally not the case that \( \text{Var}(\sum X_i) = \sum \text{Var}(X_i) \).

**EXAMPLE B** Finding the variance of a binomial random variable from the definition of variance and the frequency function of the binomial distribution is not easy (try it). But expressing a binomial random variable as a sum of independent Bernoulli random variables makes the computation of the variance trivial. Specifically, if \( Y \) is a binomial random variable, it can be expressed as \( Y = X_1 + X_2 + \cdots + X_n \), where the \( X_i \) are independent Bernoulli random variables with \( P(X_i = 1) = p \). We saw earlier (Example A in Section 4.2) that \( \text{Var}(X_i) = p(1-p) \), from which it follows from Corollary B that \( \text{Var}(Y) = np(1-p) \).

**EXAMPLE C** Random Walk

A drunken walker starts out at a point \( x_0 \) on the real line. He takes a step of length \( X_1 \), which is a random variable with expected value \( \mu \) and variance \( \sigma^2 \), and his position