How do connectivity/spectral properties of a random graph change when adding an edge? We show that a constant fraction of nonedges are such that their addition decreases the spectral gap. The result hinges on a new delocalization result for the 2nd eigenvector.

Braess’s paradox

The addition of an extra road in a traffic network can increase the overall journey time for all drivers.

Adding an edge to random graphs

A constant fraction of edges of $G(n,p)$ are such that their removal increases the spectral gap.

For a graph $G = (V,E)$, let

\[ a_G = \text{fraction of nonedges in } G \text{ whose addition decreases } \lambda_2, \]

i.e., for which

\[ \lambda_2 (L_{G_u}) < \lambda_2 (L_G), \]

Theorem. Let $p \in (0,1)$ be fixed. There exists a constant $c = c(p) > 0$ such that

\[ \mathbb{P} \left( a_G (G(n,p)) \geq c \right) \to 1. \]

Moreover, one can take $c = 1/2 - \delta$ for any constant $\delta > 0$.

Comments:

- The optimal constant should be $c = 1/2 - \delta$. See below for more.
- Normalization is key to this phenomenon. For the combinatorial Laplacian $L = D - A$ adding an edge always increases the spectral gap.
- Removing an edge: the results should be similar, but the proof of the one above is asymmetric. See below for more.
- Sparse graphs: Our proofs show that the results hold for $p = n^{-\epsilon}$ for some $\epsilon > 0$. We did not try to optimize the dependence on $p$.

Proof ideas

The first eigenvector of $L_G$ is $D^{1/2}f_2$. Let $f_2$ be the 2nd eigenvector of $L_G$.

Idea: consider $f_3$ as the 2nd eigenvector of $L_{G_t}$.

\[ \lambda_2 (L_{G_t}) = \min_{x: x \perp D^{1/2}f_2, \ l^2 x} \frac{\sum_{u \neq v} L_G(x_u - x_v)^2}{l^2} \leq \frac{32}{\sqrt{n}} \mathbb{E}_{x|D} f_3 x + 8 \frac{\mathbb{E}_{x|D} (f_2(x))^2}{(np)^2} \leq f_2(u) f_2(v) \]

This gives a general sufficient condition for the spectral gap to decrease. Specialized to $G(n,p)$ this becomes

\[ \frac{32}{\sqrt{n}} \mathbb{E}_{x|D} f_2(x)^2 + \frac{8}{(np)^2} \leq f_2(u) f_2(v). \]

The result above then follows from the following delocalization result:

Theorem. Fix $p \in (0,1)$. Let $f_2$ be the 2nd eigenvector of $L_{G(n,p)}$ with unit norm. For every $\delta > 0$ there exists a constant $C = C(p, \delta)$ such that

\[ \mathbb{P} \left( \frac{1}{n} \max \left\{ \| f_2(x) \| : x \perp D^{1/2}f_2, \ l^2 x \right\} \geq \frac{1}{\sqrt{n} (\log (n))^{C}} \right) \geq 1 - \delta \to 1. \]

Future directions

- Full delocalization. This would imply the optimal constant in the main theorem.
- Sparse graphs. What happens when $p$ goes to 0 with $n$?
- Removing an edge.
- How do other notions of connectivity/mixing change?

Acknowledgements

We are grateful to Fan Chung and the Simons Institute at UC Berkeley. This work is supported by NSF grant DMS 1106999 (M.Z.R.), and by an NSF Graduate Research Fellowship, grant no. DGE 1106400 (T.S).

References