Solutions to Miscellaneous Problems from Probability by J. Pitman Stat 20 Sec. 1 (Pitman) Fall 2005

1.rev.1. $P(\text{both defective} \mid \text{item picked at random defective})$ = $\frac{P(\text{both defective})}{P(\text{item picked at random defective})} = \frac{3\%}{3\%+5\%\times1/2} = \frac{6}{11}$

1.rev.2.

From the above tree diagram, P(black) = P(white) = 1/2, P(black, black) = P(white, white) = 1/3, P(black, white) = P(white, black) = 1/6.So $P(\text{white} \mid \text{at least one of two balls drawn was white}) = \frac{1/6+1/3}{1/6+1/6+1/3} = 3/4.$

1.rev.3. False. Of course P(HHH or TTT) = 1/4. The problem with the reasoning is that while two of the coins at least must land the same way, which two is not known in advance. Thus given say two or more H's, i.e. HHH, HHT, HTH, or THH, these 4 outcomes are equally likely, so $P(HHH \text{ or } TTT \mid \text{at least two H's}) = 1/4$, not 1/2. Similarly given at least two T's.

1.rev.4. a) P(black | Box 1) = 3/5 = P(black | Box 2)and P(red | Box 1) = 2/5 = P(red | Box 2). Hence P(black) = P(black | Box i) and P(red) = P(red | Box i) for i = 1, 2, and so the color of the ball is independent of which box is chosen. Or you can check that P(black, Box 1) = P(black)P(Box 1), etc, from the following table.

	Box 1	Box 2	
black	3/10	3/10	3/5
red	1/5	1/5	2/5
	1/2	1/2	

b)

	Box 1	Box 2	
black	3/10	5/18	26/45
red	1/5	2/9	19/45
	1/2	1/2	

Observe that $P(\text{black}, \text{Box } 1) \neq P(\text{black})P(\text{Box } 1)$, so color of ball is not independent of which box is chosen.

- a) The probability that there will be no one in favor of 134 is the probability that the 1.rev.7. 1st person doesn't favor 134 and the 2nd person doesn't favor 134 and the 3rd person doesn't favor 134 and the 4th person doesn't favor 134. This is just $\frac{20}{50} \times \frac{19}{49} \times \frac{18}{48} \times \frac{17}{47} = .021$
 - b) The probability that at least one person favors 134 is 1- the probability that no one favors 134; but the probability that no one favors 134 was done in part a). Thus the answer is 1 - .021 = .979.
 - c) The probability that exactly one person favors 134 is $\binom{4}{1}$ times the probability that the 1st person favors and the 2nd person doesn't and the 3rd person doesn't and the 4th person doesn't. This is $\binom{4}{1} \times \frac{30}{50} \times \frac{19}{49} \times \frac{18}{48} \times \frac{17}{47} = 0.126$.
 - d) The probability that a majority favor 134 is the probability that three people favor 134 plus the probability that four people favor 134. This probability is $\binom{4}{1} \times \frac{30}{50} \times \frac{29}{49} \times \frac{28}{48} \times \frac{20}{47} + \frac{30}{50} \times \frac{29}{49} \times \frac{28}{48} \times \frac{27}{47} = 0.472$

1.rev.9. a) By independence,
$$P(A \text{ and } B \text{ and } C) = P(A) \cdot P(B) \cdot P(C) = \frac{1}{60} = .01666...$$

- b) $P(A \text{ or } B \text{ or } C) = 1 P(A^c B^c C^c)$ = $1 P(A^c) \cdot P(B^c) \cdot P(C^c) = 1 \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{3}{5} = .6.$ Or use inclusion-exclusion.
- c) $P(\text{exactly one of the three events occurs}) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC))$ $= \frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{26}{60} = \frac{13}{30} = .433...$
- **2.rev.1.** a) $\binom{10}{4}(1/6)^4(5/6)^6$ b) $\binom{10}{4}(1/5)^4(4/5)^6$ c) $\frac{10!}{4!3!2!}/6^{10}$
 - d) $\frac{\binom{6}{3}}{\binom{10}{2}} = \frac{\binom{7}{4}}{\binom{10}{1}}$

2.rev.3. a)
$$P(3H) = P(3H|3 \text{ spots})P(3 \text{ spots}) + \dots + P(3H|6 \text{ spots})P(6 \text{ spots})$$

 $= \left\{ \binom{3}{3}(1/2)^3 + \binom{4}{3}(1/2)^4 + \binom{5}{5}(1/2)^5 + \binom{6}{3}(1/2)^6 \right\} \times \frac{1}{6}$
 $= \left(\frac{1}{2^3} + \frac{4}{2^4} + \frac{10}{2^5} + \frac{20}{2^6} \right) \times \frac{1}{6} = \frac{1}{6}.$
b)
 $P(4 \text{ spots}|3H) = \frac{P(3H|4 \text{ spots})P(4 \text{ spots})}{P(3H)} = \frac{\binom{4}{3}(1/2)^4(1/6)}{(1/6)} = \frac{1}{4}$

 $\overline{P(3H)}$

(1/6)

2.rev.4. P(exactly 9 tails | at least 9 tails)= P(exactly 9 tails and at least 9 tails | at least 9 tails)= P(exactly 9 tails | at least 9 tails)= $\frac{10(1/2)^{10}}{10(1/2)^{10} + (1/2)^{10}} = 10/11$

2.rev.5.
$$\frac{\binom{97}{57}}{\binom{100}{60}}$$

2.rev.6. a) $(7/10)^4 - (6/10)^4$

b) The six must be one of the four numbers drawn. The remaining three numbers must be selected from $\{0, ..., 5\}$. The desired probability is therefore $\binom{6}{3}/\binom{10}{4}$.

2.rev.7. $k \approx 1025$

2.rev.8.
$$\sum_{k=0}^{10} \left[\binom{10}{k} (1/6)^k (5/6)^{10-k} \right]^2$$

2.rev.9. a) 80%

b) P(# of kids = x | at least 2 girls)

$$= \frac{0.4 \times \frac{1}{4}}{P(\ge 2 \text{ girls})} \text{ for } x = 2$$
$$= \frac{0.3 \times \frac{4}{8}}{P(\ge 2 \text{ girls})} \text{ for } x = 3$$
$$= \frac{0.1 \times \frac{11}{16}}{P(\ge 2 \text{ girls})} \text{ for } x = 4$$

So x = 3 is most likely.

c) P(1G,3B & pick G) + P(2G,2B & pick G) + P(3G,1B & pick G)

$$=\frac{4}{16}\cdot\frac{1}{4}+\frac{6}{16}\cdot\frac{2}{4}+\frac{4}{16}\cdot\frac{3}{4}=0.4375$$

2.rev.15. a) $\binom{20}{5}(0.4)^5(0.6)^{15}$

- b) $\frac{20!}{2!4!6!8!}(0.1)^2(0.2)^4(0.3)^6(0.4)^8$
- c) P(25th ball is red, and there are 2 red balls in first 24 draws)

$$= 0.1 \times \binom{24}{2} (0.1)^2 (0.9)^{22}$$

2.rev.16. a) $\frac{\binom{48}{4}}{\binom{52}{8}}$ b) $\frac{1}{\binom{52}{8}}$ c) $13 \times \frac{\binom{48}{4}}{\binom{52}{8}} - \binom{13}{2} \times \frac{1}{\binom{52}{8}}$ **3.rev.1.** a) $1 - (5/6)^{10}$ b) 10/6 c) 35 d) $\frac{\binom{42}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$ Stat 20 Sec. 1 (Pitman) Fall 2005: Solutions to Miscellaneous Problems from Probability by J. Pitman4

e)
$$\frac{1}{2} \left(1 - P(\text{same number of sixes in first five rolls as in second five rolls})\right)$$

= $\frac{1}{2} \left\{ 1 - \sum_{k=0}^{5} \left[{5 \choose k} (1/6)^k (5/6)^{5-k} \right]^2 \right\}$

3.rev.2. a)

 $P(\text{first 6 before tenth roll}) = P(\text{at least one 6 in first 9 rolls}) = 1 - (5/6)^9 = .806194$

b)

P(third 6 on tenth roll) = P(two sixes on first nine rolls, 6 on tenth roll)

$$= \binom{9}{2} (1/6)^2 (5/6)^7 (1/6) = .046507$$

c)

P(three 6's in first ten rolls | six 6's in first twenty rolls)

= P(three 6's in first ten rolls, three sixes in last ten rolls)/P(six 6's in first twenty rolls)

$$=\frac{\binom{10}{3}(1/6)^3(5/6)^7\binom{10}{3}(1/6)^3(5/6)^7}{\binom{20}{6}(1/6)^6(5/6)^{14}}=\frac{\binom{10}{3}^2}{\binom{10}{6}}=.371517.$$

- d) Want the expectation of the sum of six geometric (1/6) random variables, each of which has expectation 6. So answer: 36.
- e) Coupon collector's problem: the required number of rolls is 1 plus a geometric (5/6) plus a geometric (4/6) plus etc. up to a geometric (1/6), so the expectation is

$$1 + (6/5) + (6/4) + (6/3) + (6/2) + (6/1) = 14.7.$$

3.rev.3. $X : \max(D_1, D_2) Y = \min(D_1, D_2)$

$$P(X = x) = P(X \le x) - P(X \le x - 1) = \left(\frac{x}{6}\right)^2 - \left(\frac{x - 1}{6}\right)^2 = \frac{2x - 1}{36} (x = 1, \dots, 6)$$

$$P(Y = y, X = 3) = \begin{cases} \frac{2}{36} & \text{for } y = 1, 2\\ \frac{1}{36} & \text{for } y = 3\\ 0 & \text{else} \end{cases}$$
So $P(Y = y \mid X = 3) = \begin{cases} \frac{2}{5} & \text{for } y = 1, 2\\ \frac{1}{5} & \text{for } y = 3\\ 0 & \text{else} \end{cases}$

$$P(X = x \mid Y = y) = \begin{cases} \frac{2}{36} & \text{for } 1 \le y < x \le 6\\ \frac{1}{36} & \text{for } 1 \le y \le x \le 6\\ 0 & \text{else} \end{cases}$$

 $E(X+Y) = E(D_1 + D_2) = 7$

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- **3.rev.6.** a) Let Y be the number of times the gambler wins in 50 plays. Then Y has binomial (50, 9/19) distribution, and $P(\text{ahead after 50 plays}) = P(Y > 25) = \sum_{i=26}^{50} {50 \choose i} (9/19)^i (10/19)^{50-i}.$ Note that the gambler's capital, in dollars, after 50 plays is 100 + 10Y + (-10)(50 - Y) = 220Y - 400. So $P(\text{not in debt after 50 plays}) = P(20Y - 400 > 0) = P(Y > 20) = \sum_{i=21}^{50} {50 \choose i} (9/19)^i (10/19)^{50-i}.$
 - b) E(capital) = E(20Y 400) = 20E(Y) 400 = 20(50)(9/19) 400 = 1400/19; $Var(\text{capital}) = Var(20Y - 400) = 400Var(Y) = 400(50)(9/19)(10/19) = 1800000/19^2$.
 - c) Use the normal approximation to the binomial: E(Y) = 23.7, and SD(Y) = 3.53, so $P(Y > 25) \approx 1 - \Phi\left(\frac{25.5 - 23.7}{3.53}\right) = 1 - \Phi(.51) = .305$. $P(Y > 20) \approx 1 - \Phi\left(\frac{20.5 - 23.7}{3.53}\right) = 1 - \Phi(-.91) = .8186$.

3.rev.7. a) $P(\geq 4 \text{ heads in 5 tosses}) = \frac{6}{32} = 0.1875$ b) $P(0, 1, \text{ or 2 heads in 5 tosses}) = \frac{16}{32} = 0.5$ c) $\frac{\text{poss. vals}}{\text{probs}} \begin{vmatrix} 0 & 1 & 2 \\ \frac{26}{32} & \frac{5}{32} & \frac{1}{32} \end{vmatrix} EX = 0.21875$ **3.rev.25.** a) $\frac{z}{P(Y_1 + Y_2 = z)} \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ P(Y_1 + Y_2 = z) & \frac{9}{36} & \frac{12}{36} & \frac{10}{36} & \frac{4}{36} & \frac{1}{36} & \frac{1}{36} & \frac{10}{36} & \frac{1}{36} & \frac{1}$

c) One possibility:
$$f(x) = \begin{cases} 1 & \text{if } x = 4,5 \\ 2 & \text{if } x = 6 \end{cases}$$

6.rev.14. Pick a person at random from the population. Let H be that person's height, and let G (= m, f) be the gender of that person.

a)
$$E(H) = E(H|G = m)P(G = m) + E(H|G = f)P(G = f) = 67 \cdot (1/2) + 63 \cdot (1/2) = 65$$
.
b)

$$E(H^2) = E(H^2|G = m)P(G = m) + E(H^2|G = f)P(G = f)$$

= $(3^2 + 67^2) \cdot (1/2) + (3^2 + 63^2) \cdot (1/2)$
= 4238

so $Var(H) = 4238 - (65)^2 = 13$. Or use $Var(H) = E[Var(H|G)] + Var[E(H|G)] = (3^2 + 3^2) \cdot (1/2) + (\frac{67 - 63}{2})^2 = 9 + 4 = 13.$ c) $P(63 \le H \le 67) = P(63 \le H \le 67|G = m)P(G = m) + P(63 \le H \le 67|G = f)P(G = f)$

$$P(63 < H < 67) = P(63 < H < 67|G = m)P(G = m) + P(63 < H < 67|G = f)P(G = f)$$
$$= (1/2)P\left(\frac{63 - 67}{3} < Z < \frac{67 - 67}{3}\right) + (1/2)P\left(\frac{63 - 67}{3} < Z < \frac{67 - 63}{3}\right)$$
$$= \Phi(4/3) - \Phi(0)$$
$$= .4088$$

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d) $P(63 < H < 67) = \Phi\left(\frac{67-65}{\sqrt{13}}\right) - \Phi\left(\frac{63-65}{\sqrt{13}}\right) = .4209.$ The answers are different because in c), the distribution of H is *not* normal: it's a mixture

The answers are different because in c), the distribution of H is *not* normal: it's a mixture of normals, which results in a normal distribution only if the two distributions being mixed are identical. The answers to c) and d) differ only slightly because the distributions in c) and d) are both unimodal and symmetric about 65, with the same standard deviation, and we are seeking the probability of an event which is symmetric about 65.

Remark: You can show by calculus that a half-and-half mixture of a normal (a, σ^2) distribution with a normal (b, σ^2) distribution will have a bimodal density if and only if $|a - b| > 2\sigma$.

e) Let X be the man's height, and Y the woman's height. Argue that X - Y = X + (-Y) has normal distribution with mean 67 - 63 = 4, and variance $3^2 + 3^2 = 18$, and therefore

$$P(X > Y) = P(X - Y > 0) = P\left(\frac{X - Y - 4}{\sqrt{18}} > \frac{-4}{\sqrt{18}}\right) = \Phi(4/\sqrt{18}) \approx .83$$