Setup: \(X_1, X_2, \cdots\) i.i.d. according to some \(F=\)"theoretical distribution"

**Definition 22.1** \(F_n(x, \omega) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i(\omega) \leq x)\)

Assume for simplicity that \(F\) is continuous.

Look at the Kolmogorov-Smirnov statistic:

\[ D_n := \sqrt{n} \sup_x |F_n(x) - F(x)| \]

Then by CLT for Binomial case

\[ \sqrt{n} \sup_x |F_n(x) - F(x)| \overset{d}{\rightarrow} N(0, F(x)[1 - F(x)]) \]

for each \(x\).

What about as \(x\) varies?

Take \(x, y\): we get two correlated Gaussian by application of multidimensional CLT.

So we expect there is a limiting Gaussian process, s.t.

\[ \sqrt{n} \sup_x |F_n(x) - F(x)| \overset{d}{\rightarrow} G(x) \]

A simplification:

We really don’t need to deal with \(F(x)\). Transform the data by

\[ X_i \rightarrow F(X_i) \overset{\text{def}}{=} U_i \]

Then

1) \(U_1, U_2, \cdots\) are i.i.d. and in \([0, 1]\)

2) \(D_n(X_1, X_2, \cdots, X_n, F) = D_n(U_1, U_2, \cdots, U_n, U[0, 1])\)

So we may as well work with \(U_1, U_2, \cdots\) i.i.d. and in \([0, 1]\), i.e.

\[ D_n = \sup_t |H_n(t)| \]

where

\[ H_n(t) := \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(U_i \leq t) - t \right) \]
Brownian Bridge

And

\[ EH_n(t) = 0, \quad Var H_n(t) = t(1 - t) \]

\[
E(H_n(s)H_n(t)) = Cov(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} 1(U_i \leq s), \frac{1}{\sqrt{n}} \sum_{j=1}^{n} 1(U_j \leq t))
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(\frac{1}{\sqrt{n}} 1(U_i \leq s), \frac{1}{\sqrt{n}} 1(U_j \leq t))
\]

\[
= Cov(\frac{1}{\sqrt{n}} 1(U_i \leq s), \frac{1}{\sqrt{n}} 1(U_j \leq t))
\]

\[
= s(1 - t)
\]

**Summary:** The FDD’s of \((H_n(t), 0 \leq t \leq 1)\) \(\xrightarrow{d}\) FDD’s of \((B^0(t), 0 \leq t \leq 1)\) where \(B^0(t)\) is a Gaussian process with \(EB^0(t) = 0, \quad E(B^0(s)B^0(t)) = s(1 - t)\)

**Theorem 22.1** There exits a version of this Gaussian process with continuous path. Moreover, such a process can be constructed in various ways for Brownian motion \(B\):

\[ B^0(t) = B(t) - tB(1) \], \quad 0 \leq t \leq 1

and \((B^0(t), 0 \leq t \leq 1)\) and \(B(1)\) are independent.

**Proof Sketch:** It’s easy to check that

\[ EB^0(t) = 0 \]

\[ EB^0(s)B^0(t) = s(1 - t) \] for \(0 < s < t\)

using \(EB(s) = 0, \quad EB(s)B(t) = s \wedge t\).

\[ E(B^0(s)B(1)) = 0 \]

\[ \Rightarrow E(\sum_{i} a_i B^0(s_i))B(1)) = 0 \]

\[ \Rightarrow B(1) \text{ is independent of } B^0(s), \quad s \in \text{finite set} \]

\[ \Rightarrow B(1) \text{ is independent of } B^0(s) \text{ for } 0 \leq s \leq 1 \]

**Theorem 22.2** \(D_n = \sup_{0 \leq t \leq 1} |H_n(t)| \xrightarrow{d} \sup_{0 \leq t \leq 1} |B^0(t)|\)

**Remark:** \(B^0\) should be understood informally as \(B|B(1) = 0\).

Rigorously, \((B(t), 0 \leq t \leq 1||B(1)| < \epsilon) \xrightarrow{d} (B^0(t), 0 \leq t \leq 1)\) in the first instance. This is for FDD’s. It’s also true in \(C[0, 1]\).

**Remark2:** It’s also true that

\[ (H_n(t), 0 \leq t \leq 1) \xrightarrow{d} (B^0(t), 0 \leq t \leq 1) \]

in sense of FDD’s. But it’s false in \(C[0,1]\) for a trivial reason \(H_n(.) \) has jumps!

What can you do?

\[ ^1 \text{So } E(B^0(s))^2 = s(1 - s) \approx s \text{ as } s \approx 0 \]
Definition 22.2 \( D[0,1] := \text{space of path which is right-continuous with left limits.} \)

Put a suitable topology. Then get \( \mathbb{d} \rightarrow \) for process with paths in \( D[0,1] \).

Proof Sketch:\(^2\)

\( \sup_{0 \leq t \leq 1} H_n(t) \) is a function of the order statistic \( U_{n,1}, U_{n,2}, \ldots, U_{n,n} \) of \( U_1, U_2, \ldots, U_n \)

\[
\sup_{0 \leq t \leq 1} H_n(t) = \text{max of } n \text{ values computed from } U_{n,1}, U_{n,2}, \ldots, U_{n,n}
\]

General tool for handling formula for functions of \( U_{n,1}, U_{n,2}, \ldots, U_{n,n} \):

Let \( w_1, \ldots, w_n \) be i.i.d. with \( P(w_i > t) = e^{-t} \) and let \( z_n = w_1 + \cdots + w_n \). Then

\[
(U_{n,1}, U_{n,2}, \ldots, U_{n,n}) \overset{d}{=} \left( \frac{z_1}{z_{n+1}}, \ldots, \frac{z_n}{z_{n+1}} \right)
\]

Now \( \sup_{0 \leq t \leq 1} H_n(t) \overset{d}{=} \text{some function of } z_1, \ldots, z_{n+1} \). Apply Dunker’s theorem to \( z_1, z_2, \ldots \) to deduce

\[
\sup_{0 \leq t \leq 1} H_n(t) \overset{d}{=} \sup_{0 \leq t \leq 1} B^0(t)
\]

and \( B^0(t) = B(t) - tB(1) \).

Alternate construction of \( B^0(t)\):

Let \( \hat{B}(t) = h(t)B(\frac{t}{1-t}) \).

Idea: make paths of \( \hat{B}(t) \overset{d}{=} B^0(t) \) by using a space time change

\[
[0,1] \longleftrightarrow [0,\infty)
\]

\[
t \rightarrow \frac{t}{1-t}
\]

Now choose \( h(t) \) s.t. \( \hat{B}(t) \overset{d}{=} B^0(t) \).

Check the variance:

\[
h^2(t) \frac{t}{1-t} = t(1-t) \Rightarrow h(t) = 1-t
\]

Prove that it’s a bridge:

Notice that \( 0 \leq s \leq t < 1 \Rightarrow \frac{s}{1-s} \leq \frac{t}{1-t} \). So

\[
E(\hat{B}(s)\hat{B}(t)) = E[(1-s)(1-t)B(\frac{s}{1-s})B(\frac{t}{1-t})] = (1-s)(1-t)\frac{s}{1-s} = s(1-t)
\]

Remark: This a a good transformation because it pushes lines to lines, crossings to crossings.

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\(^2\)see textbook for details