Let $Y_i$ be independent random variables, $Y_i$ with values in $\{0, 1, 2, \ldots \}$ and each $Y_i$ an indicator variable with values in $\{0, 1\}$ and $P(Y_i = 1) = 1/i$ for each $i = 1, 2, \ldots$. For $n = 1, 2, \ldots$ let

$$X_{n+1} := \max\{k : 1 \leq k < X_n \text{ and } Y_k = 1\}$$

and $X_{n+1} := 0$ if $X_n \leq 1$. Explain why $(X_n)$ is a Markov chain, and describe its state space and transition probabilities.

2. For $Y_1, Y_2, \ldots$ as in the previous question, let $T_0 := 0$ and for $n = 1, 2, \ldots$ let

$$T_n := \min\{k : k > T_{n-1} \text{ and } Y_k = 1\}$$

Explain why $(T_n)$ is a Markov chain, and describe its state space and its transition probabilities.

3. Let $X, Y, Z$ be random variables defined on a common probability space, each with a discrete distribution. Explain why the function $\psi(x) := E(Y \mid X = x)$ is characterized by the property

$$E(Y g(X)) = E[\psi(X) g(X)]$$

for every bounded function $g$ whose domain is the range of $X$. Use this characterization of $E(Y \mid X)$ to verify the formula

$$E(E(Y \mid X) \mid f(X)) = E[Y \mid f(X)]$$

for every function $f$ whose domain is the range of $X$, and the formula

$$E(E(Y \mid X, Z) \mid X) = E[Y \mid X].$$

4. Suppose that a sequence of random variables $X_0, X_1, \ldots$ and a function $f$ are such that

$$E(f(X_{n+1}) \mid X_0, \ldots, X_n) = f(X_n)$$

for every $n = 0, 1, 2, \ldots$. Explain why this implies

$$E(f(X_{n+1}) \mid f(X_0), \ldots, f(X_n)) = f(X_n).$$

Give an example of such an $f$ which is not constant for $(X_n)$ a $p \uparrow, 1 - p \downarrow$ random walk on the integers.

5. Let $S := X_1 + \cdots + X_N$ be the number of successes and $F := N - S$ the number of failures in a Poisson($\mu$) distributed random number $N$ of Bernoulli trials, where given $N = n$ the $X_1, \ldots, X_n$ are independent with $P(X_i = 1) = 1 - P(X_i = 0) = p$ for some $0 \leq p \leq 1$. Derive the joint distribution of $S$ and $F$. How can the conclusion be generalized to multinomial trials?

6. Let $P_i$ govern a $p \uparrow, 1 - p \downarrow$ walk $(S_n)$ on the integers started at $S_0 = i$, with $p > q$. Let

$$f_{ij} := P_i(S_n = j \text{ for some } n \geq 1).$$

Use results derived in lectures and/or the text to present a formula for $f_{ij}$ in each of the two cases $i > j$ and $i < j$. Deduce a formula for $f_{ij}$ for $i = j$.

7. Let $P_i$ govern $(X_n)$ as a Markov chain starting from $X_0 = i$, with finite state space $S$, and transition matrix $P$ which has a set of absorbing states $B$. Let $T := \min\{n \geq 1 : X_n \in B\}$ and assume that $P_i(T < \infty) = 1$ for all $i$. Derive a formula for

$$P_i(X_{T-1} = j, X_T = k)$$

for $i, j \in B^c$ and $k \in B$ in terms of the matrices $W := (I - Q)^{-1}$ and $R$, where $Q$ is the restriction of $P$ to $B^c \times B^c$ and $R$ is the restriction of $P$ to $B^c \times B$. 

Convention throughout: $\min \emptyset = \infty$. 


8. In the same setting, let \( f_{ij} := P_i(X_n = j \text{ for some } n \geq 1) \). For \( i, j \in B^c \), find and explain a formula for \( f_{ij} \) in terms of \( W_{ij} \) and \( W_{jj} \).

9. In the same setting, let \( \phi_i(s) \) denote the probability generating function of \( T \) for the Markov chain started in state \( i \). Derive a system of equations which could be used to determine \( \phi_i(s) \) for all \( i \in S \).

10. Let \( X \) be a non-negative integer valued random variable with probability generating function \( \phi(s) \) for \( 0 \leq s \leq 1 \). Let \( N \) be independent of \( X \) with the geometric(\( p \)) distribution \( P(N = n) = (1 - p)^n p \) for \( n = 0, 1, 2, \ldots \), where \( 0 < p < 1 \). Find a formula in terms of \( \phi \) and \( p \) for \( P(N < X) \).

11. Let \( X \) be a non-negative random variable with usual probability generating function \( \phi(s) \) for \( 0 \leq s \leq 1 \). Define the tail probability generating function \( \tau(s) \) by

\[
\tau(s) := \sum_{n=1}^{\infty} P(X \geq n)s^n
\]

Use the identity

\[
P(X = n) = P(X \geq n) - P(X \geq n + 1)
\]

to help derive a formula for \( \tau(s) \) in terms of \( s \) and \( \phi(s) \) for \( 0 \leq s < 1 \). Discuss what happens for \( s = 1 \).

12. Consider a random walk on the 3 vertices of a triangle labeled clockwise by 0, 1, 2. At each step, the walk moves clockwise with probability \( p \) and counter-clockwise with probability \( q \), where \( p + q = 1 \). Let \( P \) denote the transition matrix. Observe that

\[
P^2(0, 0) = 2pq; \quad P^3(0, 0) = p^3 + q^3; \quad P^4(0, 0) = 6p^2q^2.
\]

Derive a similar formula for \( P^5(0, 0) \).

13. A branching process with Poisson(\( \lambda \)) offspring distribution started with one individual has extinction probability \( p \) with \( 0 < p < 1 \). Find a formula for \( \lambda \) in terms of \( p \).

14. Suppose \( (X_n) \) is a Markov chain with state space \( \{0, 1, \ldots, b\} \) for some positive integer \( b \), with states 0 and \( b \) absorbing and no other absorbing states. Suppose also that \( (X_n) \) is a martingale. Evaluate

\[
\lim_{n \to \infty} P_a(X_n = b)
\]

and explain your answer carefully.