A. Warmup. Little explanation required. Just apply results from class notes or text.

1. A Poisson($\lambda$) number of dice are rolled. Let $N_i$ be the number of times face $i$ appears among these dice. Describe the joint distribution of $N_1$ and $N_2$.

2. A random walk starting at 3 moves up 1 at each step with probability $2/3$ and down 1 with probability $1/3$. What is the probability that the walk reaches 10 before 0?

3. For the same random walk, what is the expected number of steps until reaching either 0 or 10?

4. Let $T_r$ be the number of tosses until the $r$th H in independent coin tosses with $p$ the probability of H on each toss. Write down the probability generating function of $T_r$.

5. Let $U_r$ be the number of tosses until the pattern of "HTHT...HT" of length $2r$ appears (meaning the "HT" is repeated $r$ times in a row). Give a formula for $E(U_r)$.

B. Consider a Markov chain $(X_n)$ with transition matrix $P$. Let $f$ be a function with numerical values defined on the state space of the Markov chain. Using suitable matrix notation and/or summation notation,

1. For positive integers $m$ and $n$ give a formula for $E(f(X_{n+m}) \mid X_n = k)$

2. Suppose that $f(i) = \sum_j P(i, j)f(j)$ for all states $i$. What can you then say about the process $f(X_n)$? Explain carefully.

3. Suppose that $f(i) = \sum_j P(i, j)f(j)$ for all states $i$, that $|f(i)| \leq 5$ for all states $i$, that there are precisely two absorbing states 0 and $b$, one or other of which is reached in finite time with probability 1, no matter what the initial state $i$, and that $f(b) \neq f(0)$. Derive a formula for the probability, starting in state $i$, that the chain ends up being absorbed in state $b$.

C. Let $Z_n$ be the number of individuals in the $n$th generation of a branching process starting with $Z_0 = 1$, with offspring probability generating function $\phi(s) = \sum_{n=0}^{\infty} p_n s^n$. Let $N$ be the least $n$ such that $Z_n = 0$, with $n = \infty$ if no such $n$. Let $M$ be the least $n$ such that there is some individual in generation $n$ who has no children.

1. For $n = 1, 2, 3, 4$, find expressions for $P(N \geq n)$ in terms of the function $\phi$.

2. Let $q_n = P(M \geq n)$. Derive a formula for $q_{n+1}$ involving the function $\phi$, $q_n$, and $p_0$.

3. For $n = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$ find a formula for

$$P(M \geq n + 1 \mid M \geq n, Z_{n+1} = k)$$

which shows this conditional probability is at most $1 - p_0$.

4. Deduce that $E(M) \leq 1/p_0$. 

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