Statistics 134, Sec 2, Fall 2005 Prof. Pitman

FINAL EXAM

Name: SKETCH SOLUTIONS

SID number: by Prof Pitman.

SHOW CALCULATIONS, OR GIVE REASONS, ON ALL PARTS OF ALL QUESTIONS.

Numerical answers may be left unsimplified except where the question asks for a decimal or a fraction. You may refer to your text and notes. Please try not to interrupt the exam by asking questions. If you find a question is ambiguous, indicate clearly how you choose to interpret it. You may interpret it to your own advantage to best demonstrate what you have learned in the course. Good luck!

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Total	

2. A multiple choice test has 4 possible answers for each question, exactly one of which is right. The test has 20 questions. A student knows the correct answer to 14 questions and guesses at random for the other 6. Let X be the number of questions the student gets right.

a) Describe the distribution of X by a formula.

$$X = 14 + Y \quad \text{for } Y \sim \text{binomial}\left(6, \tfrac{1}{4}\right).$$

$$\mathbb{P}(X = k) = \mathbb{P}(Y = k-14)$$

$$= \binom{6}{k-14}\left(\tfrac{1}{4}\right)^{k-14}\left(\tfrac{3}{4}\right)^{6-k+14}$$

for $14 \le k \le 20$.

b) Give a numerical expression for $P(X \geq 19)$.

$$= \mathbb{P}(X = 19) + \mathbb{P}(X = 20)$$

$$= \binom{6}{5}\left(\tfrac{1}{4}\right)^5\left(\tfrac{3}{4}\right) + \binom{6}{6}\left(\tfrac{1}{4}\right)^6$$

c) Evaluate $E(X)$ as a decimal.

$$E(X) = 14 + E(Y)$$

$$= 14 + 6/4 = 15.5$$

d) Evaluate $Var(X)$ as a decimal.

$$Var(X) = Var(Y) = 6 \cdot \tfrac{1}{4} \cdot \tfrac{3}{4}$$

$$= \frac{9}{8} = 1.25$$

4. Dan and Stan each roll a six-sided die. If they roll different numbers, the one who rolled the higher number wins the difference between the numbers from the other, in dollars. If they roll the same number, they roll again until they get different numbers. As soon as they exchange money, that completes one game. They play 100 games. Let R_i be the number of times they both have to roll during the ith game, and let X_i the amount that Dan wins in the ith game. Note that $R_i \geq 1$ and that X_i can be negative, but not 0.

a) Describe the distribution of R_1.

$$P(\text{same } \#) = 6/36 = 1/6.$$

$$\Rightarrow R_1 \sim \text{geometric}(p = 5/6) \text{ on } \{1, 2, \cdots\}.$$

b) Describe the distribution of X_1.

Value	-5	-4	-3	-2	-1	1	2	3	4	5
30 * Prob	1	2	3	4	5	5	4	3	2	1

c) Find $P(R_1 + R_2 + R_3 + R_4 = 7)$.

$$R_1 + \cdot + R_4 \sim \text{neg binom}(4, 5/6).$$

$$\mathbb{P}(R_1 + \cdot + R_4 = 7) = \binom{6}{3}\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^3$$

d) Find $Var(\sum_{i=1}^{100} X_i)$.

$$= 100 \, Var(X_1)$$

$$= 100 \, \frac{2}{30}\left(5^2 \cdot 1 + 4^2 \cdot 2 + 3^2 \cdot 3 + 2^2 \cdot 4 + 1^2 \cdot 5\right).$$

e) Find the approximate value of $P(\sum_{i=1}^{100} X_i \leq 3)$.

$$\mu = 0, \quad \sigma = \sqrt{Var\left(\sum_1^{100} X_i\right)}$$

$$\rightarrow \Phi\left(\frac{3\frac{1}{2}}{\sigma}\right) \text{ by Normal Approx.}$$

6. Each vehicle arriving at a toll booth is either a car or a truck. Cars arrive as a Poisson process with rate $\lambda = 3$ per minute. Independently of the cars, trucks arrive as a Poisson process with rate $\lambda = 1$ per minute.

a) What is the probability that exactly 10 vehicles arrive in a two minute interval?

Total rate $4 = \lambda_*$, $t = 2$

$$\rightarrow \frac{e^{-\lambda_* t}(\lambda_* t)^{10}}{10!} = \frac{e^{-8}(8)^{10}}{10!}$$

b) Consider the first vehicle to arrive after time $t = 0$. What is the probability that this vehicle arrives after time $t = 1$ and is a truck?

$$e^{-4} \times \frac{1}{4}$$

c) Give a formula for the probability density of the length of time between arrivals of the 5th and 7th cars after some fixed time.

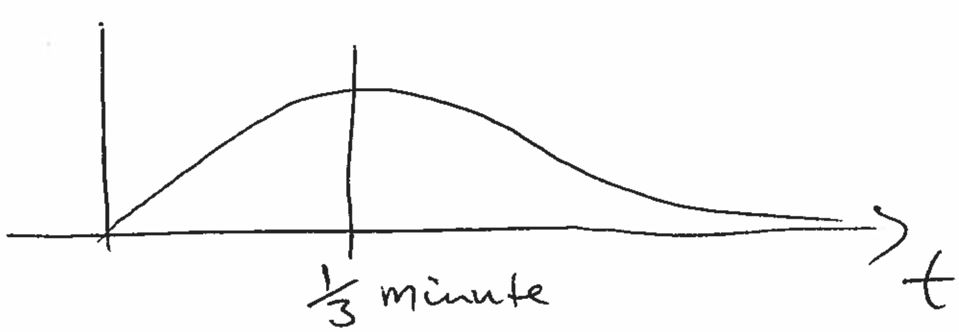
$$T_7 - T_5 \overset{d}{=} T_2$$

gamma$(2, \lambda = 3)$.

$$\text{density}(t) = \frac{1}{\Gamma(2)} \lambda^2 t e^{-\lambda t}$$

$\Gamma(2) = 1!$, $\lambda = 3$.

d) Sketch a graph of the density found in c), with a properly labeled horizontal axis.



8. A non-negative random variable X has mean 100 and variance 100.

a) Give an explicit example of a distribution of X consistent with these properties.

Poisson(100)

b) What does Markov's inequality say about $P(X \geq 400)$?

$$P(X \geq 400) \leq \frac{E(X)}{400} = \frac{100}{400} = \frac{1}{4}$$

c) What does Chebychev's inequality say about $P(X \geq 400)$?

$$P(X \geq 400) \leq P(|X - 100| \geq 300)$$
$$\leq \frac{100}{300^2} = \frac{1}{900}$$

d) Let S_n be the sum of n independent variables, each with the same distribution as X. Find a sequence x_n so that $P(S_n/n > 100 + x_n)$ converges to $1/4$ as $n \to \infty$.

Find z with $\Phi(z) = 1/4$

$$x_n = z\, SD(S_n/n)$$
$$= z \cdot 10/\sqrt{n}$$

[By CLT]

10. A random variable X has beta (a, b) distribution on the interval $[0, 1]$. Conditionally given the value of X, a series of independent trials is performed, each with success probability X and failure probability $1 - X$.

a) What is the unconditional probability that the first trial is a success?

$$P(\text{Success}) = \mathbb{E}[P(\text{success} \mid X)] = E(X) = \frac{a}{a+b}$$

b) Describe the conditional distribution of X given that the first trial is a success.

$$P(X \in dx, \text{ success}) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1} dx \cdot x$$

This is the density, up to a constant

$\Rightarrow$ ~~P(X~~ $X \mid \text{success} \sim \text{beta}(a+1, b)$

c) What is the unconditional probability of exactly k successes in the first n trials?

$$\mathbb{E} \binom{n}{k} X^k (1-X)^{n-k}$$

$$= \binom{n}{k} \frac{B(a+k, b+n-k)}{B(a,b)}.$$

d) Describe the conditional distribution of X given exactly k successes in the first n trials.

Similarly to b), is

$$\text{beta}(a+k, b+n-k).$$

e) Conditionally given exactly k successes in the first n trials, but without knowing the value of X, what is the probability of success on trial $(n + 1)$?

$$\frac{\mathbb{E} \binom{n}{k} X^k (1-X)^{n-k} X}{\mathbb{E} \binom{n}{k} X^k (1-X)^{n-k}} = \frac{B(a+k+1, b+n-k)}{B(a+k, b+n-k)}.$$