Statistics 134, Fall 2002 **Prof. Pitman**

FINAL EXAM

~~Name~~: SOLUTION SKETCH

~~SID number~~: BY Prof. Pitman.

SHOW CALCULATIONS, OR GIVE REASONS, ON ALL PARTS OF ALL QUESTIONS. DO NOT LEAVE NUMERICAL ANSWERS UNSIMPLIFIED. You may refer to your text and class materials.

1	
2	
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8	
9	
10	
Total	

1. Each time a random number generator is run, it produces a pair of digits by making two draws at random with replacement from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The generator is run 5000 times. Let X be the number of times it produces the pair 00. Find an integer n so that $P(X \geq n)$ is approximately 85%.

$$X \sim \text{binomial}\left(5{,}000, \frac{1}{100}\right)$$

$$\mu := E(X) = 50, \quad \sigma := SD(X) = \sqrt{50 \cdot \frac{99}{100}}$$

$$1 - P(X \geq n) = P(X < n)$$

$$\simeq \Phi\left(\frac{n - \frac{1}{2} - \mu}{\sigma}\right)$$

Want z so $\Phi(z) = 15\%$

$z = -1.04$ (from table)

then set

$$\frac{n - \frac{1}{2} - \mu}{\sigma} = z$$

$$n = \sigma z + \mu + \frac{1}{2}$$

3. I have three painted dice.

- The first die has one red face and five green faces.
- The second die has three red faces and three green faces.
- The third die has five red faces and one green face.

I pick one of the dice at random and roll it twice. Let R_1 be the event that the first roll shows a red face. Let R_2 be the event that the second roll shows a red face.

a) Find $P(R_1)$.

$$\frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{3}{6} + \frac{1}{3} \cdot \frac{5}{6}$$

b) Are R_1 and R_2 independent?

No

$$\mathbb{P}(R_1 R_2) = \frac{1}{3} \cdot \left(\frac{1}{6}\right)^2 + \frac{1}{3} \cdot \left(\frac{3}{6}\right)^2 + \frac{1}{3}\left(\frac{5}{6}\right)^2$$

$$\neq P(R_1)\,P(R_2).$$

Note $P(R_2) = P(R_1)$.

2. A standard deck consists of fifty-two cards. Four of the cards are aces. Cards are dealt from the deck at random without replacement until two aces have appeared.

Let X_1 be the number of cards dealt till the first ace appears. Let X_2 be the total number of cards dealt till the second ace appears. So $P(X_2 > X_1) = 1$.

a) Find $P(X_1 = 1, X_2 = 5)$.

$$\frac{4}{52} \quad \frac{48}{51} \quad \frac{47}{50} \quad \frac{46}{49} \quad \frac{3}{48}$$

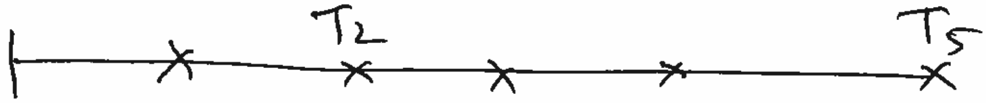
b) Find $P(X_1 = 1 \mid X_2 = 5)$.

Given $X_2 = 5$ the first ace is equally likely to be in each of the places 1, 2, 3, 4, So

$$\mathbb{P}(X_1 = 1 \mid X_2 = 5) = 1/4$$

4. Particles arrive at a Geiger counter according to a Poisson process with a rate of 2 per minute.

a) Find the expectation and standard deviation of time between the arrival of the second particle and the arrival of the fifth particle.



$$T_5 - T_2 = W_3 + W_4 + W_5 \overset{d}{=} T_3$$

$$\mathbb{E}(T_5 - T_2) = \mathbb{E}(T_3) = 3/2$$

$$SD(\cdot\cdot) = \sqrt{3}/2$$

Using formulas for gamma (r, λ), $r = 3$, $\lambda = 2$

b) Find the chance that the fifth particle arrives more than two minutes after the second particle.

$$\mathbb{P}(T_5 - T_2 > 2) = \mathbb{P}(T_3 > 2)$$

$$= \mathbb{P}(N(0,2] = 0 \text{ or } 1 \text{ or } 2)$$

$$= \sum_{i=0}^{2} e^{-\lambda t} \frac{(\lambda t)^i}{i!} \Bigg|_{\substack{t=2 \\ \lambda=2}}$$

5. Let X and Y be independent normal variables, with $E(X) = 10$, $SD(X) = 3$, $E(Y) = 20$, $SD(Y) = 5$.

Let $V = Y - X + 2$ and $W = 4X - 2Y + 4$.

a) Find the correlation between V and W.

$$\mathrm{Cov}(V, W) = E(VW) - E(V)E(W)$$
$$= \mathrm{Cov}(Y - X, 4X - 2Y)$$
$$= -2\,\mathrm{Var}(Y) - 4\,\mathrm{Var}(X)$$
$$= -2 \times 25 - 4 \times 9 = -86$$

$\mathrm{Var}(V) = 5^2 + 3^2 = 34$, $\mathrm{Var}(W) = 4^2 \cdot 3^2 - 2^2 \cdot 5^2$

$\mathrm{Corr}(V, W) = \mathrm{Cov} / \sqrt{\mathrm{Var}}\sqrt{\mathrm{Var}}$

b) Find $E(V \mid W = 25)$.

$$= E(Y) + (25 - E(W))\, \rho \frac{SD(Y)}{SD(W)}$$

where $E(V) = \cdots$

$E(W) = \cdots$

$\rho = \mathrm{Corr}(V, W)$.

c) Find $P(V > W + 20)$.

$$= P(V - W < 20)$$

Use that $V - W$ is normally distributed, and reduce to standard units.

7. A computer screen saver draws colored discs. Each disc is equally likely to be blue, green, yellow, or red, independently of all other discs. The radius of each disc (measured in inches) is chosen independently of the colors and radii of all other discs, according to the density

$$f(r) = \begin{cases} 0.5r, & 0 \leq r \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Let S be the area of the first disc drawn by the screen saver. Find the density of S. Recognize this as a named density and provide its parameters.

$S = \pi R^2$

for $s \leq \pi 2^2$

$$P(S \leq s) = P(\pi R^2 \leq s) = P\left(R \leq \sqrt{\frac{s}{\pi}}\right) = \int_0^{\sqrt{s/\pi}} \frac{1}{2} r\,dr = \left.\frac{r^2}{4}\right|_0^{\sqrt{s/\pi}} = s/4\pi, \quad 0 \leq s \leq 4\pi.$$

$$f_S(s) = \frac{d}{ds} s/4\pi = \frac{1}{4\pi}, \quad 0 \leq s \leq 4\pi$$

UNIFORM $[0, 4\pi]$

b) Find the chance that the first four discs drawn by the screen saver all have areas bigger than 10 square inches and are all of different colors.

For each disc

$$P(\text{area} > 10) = \frac{4\pi - 10}{4\pi}.$$

$$P(\text{all diff colors}) = \frac{4}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{3}{32}.$$

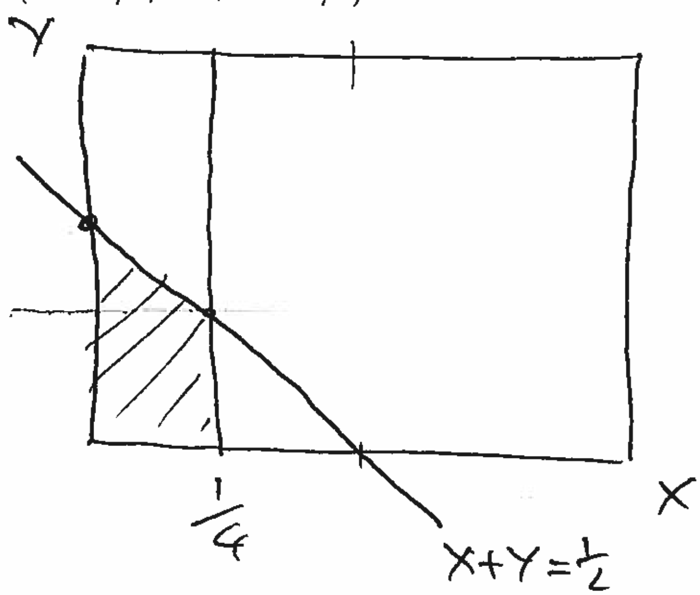
Required prob is

$$\left(\frac{4\pi - 10}{4\pi}\right)^4 \times \frac{3}{32}$$

6. Let X and Y have joint density given by

$$f(x,y) = \begin{cases} \frac{12}{11}(x^2 + xy + y^2), & 0 < x < 1, \ \ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

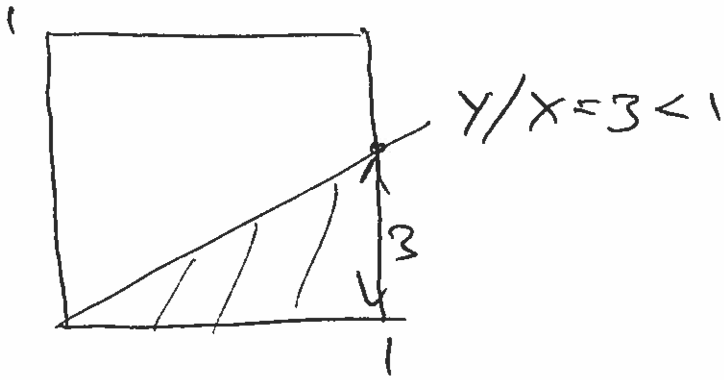
Find $P(X < 1/4,\ X + Y < 1/2)$.

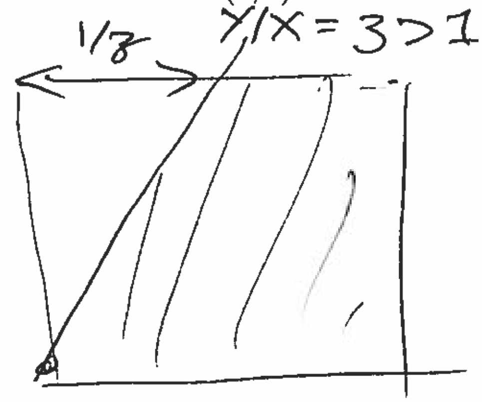


$$P(\cdots) = \int_{x=0}^{1/4} dx \int_{y=0}^{\frac{1}{2}-x} dy \ f(x,y)$$

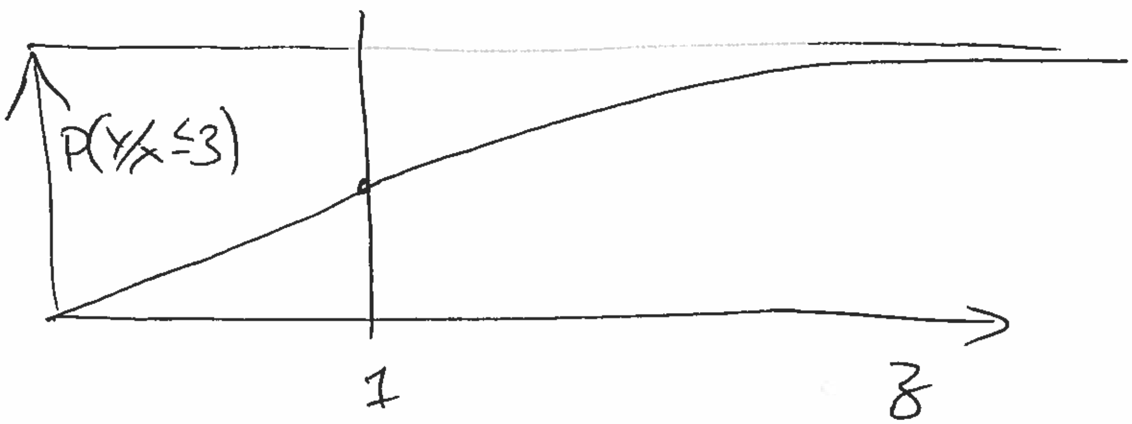
$$= \ldots$$

8. X and Y are independent and each is uniformly distributed on the interval $(0, 1)$. Find the c.d.f. of Y/X and sketch its graph.





$$P(Y/X \le z) = \begin{cases} \frac{1}{2} z & \text{if } z \le 1 \\ 1 - \frac{1}{2z} & \text{if } z > 1. \end{cases}$$



9. A survey organization is about to choose a random sample of 200 households. Suppose that for each i in the range 1 through 200, the number of residents in the ith sampled household equals $1 + N_i$, where $N_1, N_2, \ldots, N_{200}$ are independent and identically distributed Poisson variables, each with parameter 2.

Let X be the total number of residents in all the sampled households.

a) Find the expectation and standard deviation of X.

$$X = \sum_{i=1}^{200} (1 + N_i),$$

$$E(X) = 200 + 200 \times 2$$

$$SD(X) = \sqrt{200 \times 2}$$

b) Find an approximate value of $P(X > 650)$. Give a brief justification for your approximation.

$$P(X > 650) = P\left(\frac{X - E(X)}{SD(X)} > \frac{650 - E(X)}{SD(X)} \right)$$

$$\simeq 1 - \Phi\left(\frac{650 - E(X)}{SD(X)} \right),$$

by Normal Approx to Poisson with parameter 200, which is good by CLT.

c) Find the exact distribution of X.

$$X - 200 \sim \text{Poisson}(200),$$

$$\mathbb{P}(X = k) = \frac{e^{-200} (200)^{k-200}}{(k-200)!}$$

for $k \geq 200$.

10. A basket contains n balls labeled 1 to n. Near the basket there are n empty boxes labeled 1 to n. Balls are placed in the boxes as follows. A ball is picked at random from the basket and placed in Box 1. Then a ball is picked at random from the $n-1$ balls remaining in the basket, and placed in Box 2. The process continues till all n balls have been placed. At the end of the process the basket is empty and each box contains exactly one ball.

For each i from 1 to n, say that a match occurs at i if Ball i gets placed in Box i. Let M be the total number of matches.

a) Show that $E(M) = 1$.

$$M = X_1 + \cdots + X_n$$

$$X_i = 1(\text{match in box } i).$$

$$E(M) = E(X_1) + \cdots + E(X_1)$$

$$= \frac{1}{n} + \cdots + \frac{1}{n} = 1.$$

b) Show that $Var(M) = 1$.

$$E(M^2) = E(X_1 + \cdots + X_n)^2$$

$$= n \cdot \frac{1}{n} + n(n-1)\, E X_1 X_2$$

$$= 1 + n(n-1)\, \frac{1}{n} \cdot \frac{1}{n-1}$$

$$= 1 + 1.$$

$$Var(M) = E(M^2) - E(M)^2 = 1 + 1 - 1^2 = 1.$$

c) What is the approximate distribution of M when n is large? Give an intuitive justification, and show that your answer is consistent with the mean and variance of parts a) and b).

Approximately Poisson(1).

This limit dist has the same mean and variance as M_n for each n.

In the limit, the X_i are nearly independent, so M_n is nearly binomial$(n, \frac{1}{n})$

$\to$ Poisson(1)