Determinantal Processes And The IID Gaussian Power Series

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Talk based on work joint with:

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Samples of translation invariant point processes in the plane: Poisson (left), determinantal (center) and permanental for
\[ K(z, w) = \frac{1}{\pi} e^{z \bar{w} - \frac{1}{2}(|z|^2 + |w|^2)}. \] Determinantal processes exhibit repulsion, while permanental processes exhibit clumping.
Determinantal Point Processes

Let \((\Omega, \mu)\) be a \(\sigma\)-finite measure space, \(\Omega \subset \mathbb{R}^d\). One way to describe the distribution of a point process \(\mathcal{X}\) on \(\Omega\) is via its \textit{joint intensities}.

Definition: \(\mathcal{X}\) has \textit{joint intensities} \(\rho_k, k = 1, 2, \ldots\) if, for any mutually disjoint (measurable) sets \(A_1, \ldots, A_k\),

\[
E \left[ \prod_{j=1}^{k} |\mathcal{X} \cap A_j| \right] = \int \prod_{j=A_j} \rho_k(x_1, \ldots, x_k) d\mu
\]
In most cases of interest the following is valid (assume no double points):

• Ω is discrete and \( \mu = \) counting measure: \( \rho_k(x_1, \ldots, x_k) \) is the probability that \( x_1, \ldots, x_k \in \mathcal{X} \).

• Ω is open in \( \mathbb{R}^d \) and \( \mu = \) Lebesgue measure: \( \rho_k(x_1, \ldots, x_k) \) is

\[
\lim_{\varepsilon \to 0} \frac{P(\mathcal{X} \text{ has a point in each of } B_\varepsilon(x_j))}{(\text{Vol}(B_\varepsilon))^k}.
\]
Now let $K$ be the kernel of an integral operator $\mathcal{K}$ on $L^2(\Omega)$ with the spectral decomposition

$$K(x, y) = \sum_k \lambda_k \varphi_k(x) \overline{\varphi_k(y)},$$

where $\{\varphi_k\}_k$ is an orthonormal set in $L^2(\Omega)$.

**Definition:** $\mathcal{X}$ is said to be a **determinantal point process** with kernel $K$ if its joint intensities are

$$\rho_k(x_1, \ldots, x_k) = \det \left( (K(x_i, x_j))_{1 \leq i, j \leq k} \right),$$

for every $k \geq 1$ and $x_1, \ldots, x_k \in \Omega$.  

**Key facts:** (Maachi)

- A locally finite determinantal process with the Hermitian kernel $K$ exists if and only if $K$ is locally of trace class and $0 \leq \lambda_k \leq 1 \ \forall k$.

- If $K(x,y) = \sum_{k=1}^{n} \varphi_k(x)\overline{\varphi_k(y)}$, then the total number of points in $\mathcal{X}$ is $n$, almost surely. Since the corresponding integral operator $\mathcal{K}$ on $L^2(\Omega)$ is a projection, such processes are said to be **determinantal projection process**.
Karlin-McGregor (1958)

Consider $n$ independent simple symmetric random walks on $\mathbb{Z}$ started from $i_1 < i_2 < \ldots < i_n$ where all the $i_j$'s are even. Let $P_{i,j}(t)$ be the $t$-step transition probabilities.

Then the probability that at time $t$, the random walks are at $j_1 < j_2 < \ldots < j_n$ and have mutually disjoint paths is

$$
\det \begin{pmatrix}
P_{i_1,j_1}(t) & \cdots & P_{i_1,j_n}(t) \\
\vdots & \ddots & \vdots \\
P_{i_n,j_1}(t) & \cdots & P_{i_n,j_n}(t)
\end{pmatrix}.
$$

This is intimately related to determinantal processes. For instance, one can show that if $t$ is even and we also condition the walks to return to $i_1, \ldots, i_n$, then the positions of the walks at any time $s$ ($1 \leq s \leq t$) are determinantal. (See Johanson(2004) for this and more general results)
Uniform Spanning Tree

Let $G$ be a finite undirected graph. Let $T$ be uniformly chosen from the set of spanning trees of $G$. Orient the edges of $G$ arbitrarily. Let $\bar{e}$ be the opposite orientation of $e$. For each directed edge $e$, let $\chi^e := 1_e - 1_{\bar{e}}$ denote the unit flow along $e$.

$$\ell^2_-(E) = \{f : E \to \mathbb{R} : f(e) = -f(\bar{e})\}$$

$$\star = \text{span}\{\sum_{e=v} \chi^e : \text{where } v \text{ is a vertex.}\}$$

$$\diamond = \text{span}\{\sum_{i=1}^n \chi^{e_i} : e_1, \ldots, e_n \text{ is an oriented cycle}\}$$

It is easy to see that $\ell^2_-(E) = \star \oplus \diamond$. Define $I^e := P_\star \chi^e$, the orthogonal projection onto $\star$. Kirchoff (1847) proved that for any edge $e$, $P[e \in T] = (I^e, I^e)$.
Theorem: (Burton and Pemantle (1993)) The set of edges in $T$ forms a determinantal process with kernel $Y(e, f) := (I^e, I^f)$. i.e., for any distinct edges $e_1, \ldots, e_k$

$$P[e_1, \ldots, e_k \in T] = \det [(Y(e_i, e_j))_{1 \leq i, j \leq k}]. \quad (2)$$
Ginibre Ensemble

Let $A$ be an $n \times n$ matrix with i.i.d. standard complex normal entries. Then the eigenvalues of $A$ form a determinantal process in $\mathbb{C}$ with the kernel

$$K_n(z, w) = \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2 + |w|^2)} \sum_{k=0}^{n-1} \frac{(z \overline{w})^k}{k!}.$$  

As $n \to \infty$, we get a determinantal process with the kernel

$$K(z, w) = \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2 + |w|^2)} \sum_{k=0}^{\infty} \frac{(z \overline{w})^k}{k!}.$$  

$$= \frac{1}{\pi} e^{-\frac{1}{2}(|z|^2 + |w|^2) + z \overline{w}}.$$
Construction of determinantal projection processes

Define $K_H \delta_x(\cdot) = K(\cdot, x)$. The intensity measure of the process is given by

$$\mu_H(x) = \rho_1(x) d\mu(x) = \|K_H \delta_x\|^2 d\mu(x). \quad (3)$$

Note that $\mu_H(M) = \text{dim}(H)$, so $\mu_H / \text{dim}(H)$ is a probability measure on $M$. We construct the determinantal process as follows. Start with $n = \text{dim}(H)$, and $H_n = H$.

- If $n = 0$, stop.
- Pick a random point $X_n$ from the probability measure $\mu_{H_n}/n$.
- Let $H_{n-1} \subset H_n$ be the orthocomplement of the function $K_{H_n} \delta_x$ in $H_n$. In the discrete case, $H_{n-1} = \{f \in H_n : f(X_n) = 0\}$. Note that $\text{dim}(H_{n-1}) = n - 1$ a.s.
- Decrease $n$ by 1 and iterate.
Proposition: The points \((X_1, \ldots, X_n)\) constructed by this algorithm are distributed as a uniform random ordering of the points in a determinantal process \(\mathcal{X}\) with kernel \(K\).

Proof: Let \(\psi_j = K_H \delta_{x_j}\). Projecting to \(H_j\) is equivalent to first projecting to \(H\) and then to \(H_j\), and it is easy to check that \(K_{H_j} \delta_{x_j} = K_{H_j} \psi_j\). Thus, by (3), the density of the random vector \((X_1, \ldots, X_n)\) constructed by the algorithm equals

\[
p(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{\|K_{H_j} \psi_j\|^2}{j}.
\]

Note that \(H_j = H \cap \langle \psi_{j+1}, \ldots, \psi_n \rangle^\perp\), and therefore \(V = \prod_{j=1}^{n} \|K_{H_j} \psi_j\|\) is exactly the repeated “base times height” formula for the volume of the parallelepiped determined by the vectors \(\psi_1, \ldots, \psi_n\) in the finite-dimensional vector space \(H \subset L^2(M)\). It is well-known that \(V^2\) equals the determinant of the Gram matrix whose \(i, j\) entry is given by the scalar product of
\( \psi_i, \psi_j \), that is \( \int \psi_i \overline{\psi}_j d\mu = K(x_i, x_j) \). We get

\[
p(x_1, \ldots, x_n) = \frac{1}{n!} \det(K(x_i, x_j)),
\]

so the random variables \( X_1, \ldots, X_n \) are exchangeable. Viewed as a point process, the \( n \)-point joint intensity of \( \{X_j\}_j=1^n \) is \( n!p(x_1, \ldots, x_n) \), which agrees with that of the determinantal process \( \mathcal{X} \). The claim now follows since \( \mathcal{X} \) contains \( n \) points almost surely.
We have the following remarkable fact that connects the kernel $K$ to the distribution of $\mathcal{X}$:

**Theorem:** (Shirai-Takahashi (2002)) Suppose $\mathcal{X}$ is a determinantal process on $E$ with kernel $K(x, y) = \sum_k \lambda_k \varphi_k(x) \overline{\varphi}_k(y)$. Then

$$L(\mathcal{X}) = \sum_{S \subseteq \mathbb{N}} \alpha(S) L(\mathcal{X}(S)), \quad (4)$$

where $\mathcal{X}(S)$ is the determinantal process in $E$ with kernel $\sum_{j \in S} \varphi_j(x) \overline{\varphi}_j(y)$ and

$$\alpha(S) = \prod_{j \in S} \lambda_j \prod_{j \not\in S} (1 - \lambda_j).$$

In particular the number of points in the process $\mathcal{X}$ has the distribution of a sum of independent Bernoulli($\lambda_k$) random variables.
Proof: (HKPV) Assume $K$ has finite rank i.e., take

$$K(x, y) = \sum_{k=1}^{n} \lambda_k \varphi_k(x) \overline{\varphi}_k(y).$$

Otherwise we can approximate by finite rank kernels, and deduce the same for general $K$ since the corresponding processes increase (stochastically) to the original process.

Let $I_k$, $1 \leq k \leq n$ be independent Bernoulli random variables with $I_k \sim \text{Bernoulli}(\lambda_k)$. Then set

$$K_I(x, y) = \sum_{k=1}^{n} I_k \varphi_k(x) \overline{\varphi}_k(y).$$

$K_I$ is a random analogue of the kernel $K$. We want to prove $\forall m, x_i$’s,

$$E \left[ \det \begin{pmatrix} (K_I(x_i, x_j))_{1 \leq i, j \leq m} \end{pmatrix} \right] = \det \begin{pmatrix} (K(x_i, x_j))_{1 \leq i, j \leq m} \end{pmatrix}. \quad (5)$$
Proof of (5): Take $m = n$ first. Then we write

\[
\begin{pmatrix}
K_I(x_1, x_1) & \ldots & \ldots & K_I(x_1, x_n) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
K_I(x_n, x_1) & \ldots & \ldots & K_I(x_n, x_n)
\end{pmatrix} =
\begin{pmatrix}
I_1 \varphi_1(x_1) & \ldots & \ldots & I_n \varphi_n(x_1) \\
I_1 \varphi_1(x_2) & \ldots & \ldots & I_n \varphi_n(x_2) \\
\vdots & \ddots & \ddots & \vdots \\
I_1 \varphi_1(x_n) & \ldots & \ldots & I_n \varphi_n(x_n)
\end{pmatrix}
\begin{pmatrix}
\overline{\varphi}_1(x_1) & \ldots & \overline{\varphi}_1(x_n) \\
\overline{\varphi}_2(x_1) & \ldots & \overline{\varphi}_2(x_n) \\
\vdots & \ddots & \vdots \\
\overline{\varphi}_n(x_1) & \ldots & \overline{\varphi}_n(x_n)
\end{pmatrix}.
\]

Hence $\det (\left( K_I(x_i, x_j) \right)_{1 \leq i, j \leq n} = I_1 \ldots I_n \det(A^* A)$ where $A$ is the second matrix on the right side above. On taking expectations we get

\[
\mathbb{E} \left[ \det \left( (K_I(x_i, x_j))_{1 \leq i, j \leq n} \right) \right] = \lambda_1 \ldots \lambda_n \det(A^* A).
\]
Now we also have
\[
\begin{pmatrix}
K(x_1, x_1) & \ldots & \ldots & K(x_1, x_n) \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
K(x_n, x_1) & \ldots & \ldots & K(x_n, x_n)
\end{pmatrix}
= 
\begin{pmatrix}
\lambda_1 \varphi_1(x_1) & \ldots & \lambda_n \varphi_n(x_1) \\
\lambda_1 \varphi_1(x_2) & \ldots & \lambda_n \varphi_n(x_2) \\
\vdots & \ddots & \vdots \\
\lambda_1 \varphi_1(x_n) & \ldots & \lambda_n \varphi_n(x_n)
\end{pmatrix}
\begin{pmatrix}
\bar{\varphi}_1(x_1) & \ldots & \bar{\varphi}_1(x_n) \\
\bar{\varphi}_2(x_1) & \ldots & \bar{\varphi}_2(x_n) \\
\vdots & \ddots & \vdots \\
\bar{\varphi}_n(x_1) & \ldots & \bar{\varphi}_n(x_n)
\end{pmatrix}.
\]

From this we get
\[
\det \left( (K(x_i, x_j))_{1 \leq i, j \leq n} \right) = \lambda_1 \ldots \lambda_n \det(A^* A).
\]
This proves that the two point processes $\mathcal{X}$ (determinantal with kernel $K$) and $\mathcal{X}_I$ (determinantal with kernel $K_I$) have the same $n$-point joint intensity.

But both these processes have at most $n$ points. Therefore for every $m$, the $m$-point joint intensities are determined by the $n$-point joint intensities (zero for $m > n$, got by integrating for $m < n$).

This proves the theorem.
Zeros of the i.i.d. Gaussian power series [Virág-P.].

Let

\[ f_U(z) = \sum_{n=0}^{\infty} a_n z^n \]

\[ Z_U = \text{zeros}(f_U) \] (6)

with \( \{a_n\} \) complex Gaussian, density(\( re^{i\theta} \)) = \( e^{-r^2} \).

**Theorem:** (Hannay, Zelditch-Shiffman, ...)

Law of \( Z_U \) invariant under Möbius transformations \( z \rightarrow e^{i\alpha} \frac{z-\lambda}{1-\lambda z} \) that preserve unit disk.
Euclidean analog:

\[ f_C = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}}, \]  

\( \text{(7)} \)

satisfies

\[
\text{Cov}[f_C(z), f_C(w)] = E \left[ \sum_n a_n z^n \cdot \sum_k \bar{a}_k \bar{w}^k \right] = \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{n!} = e^{zw}.
\]

Thus,

\[
\text{Cov}[f_C(z + a), f_C(w + a)] = e^{(z+a)(\bar{w}+\bar{a})}
\]

\[
= \text{Cov} \left[ e^{\frac{|a|^2}{2}} e^{\bar{a}z} f_C(z), e^{\frac{|a|^2}{2}} e^{\bar{a}w} f_C(w) \right].
\]

Since Gaussian processes are determined by \( \text{Cov}(\cdot, \cdot) \) this proves translation invariance of \( \text{Law}[\text{zeros}(f_C)] \).
**Definition:** Let $p_\epsilon(z_1, \ldots, z_n)$ denote the probability that a random function $f$ has zeros in $B_\epsilon(z_1), \ldots B_\epsilon(z_n)$. **Joint intensity** of zeros (if it exists) is defined to be

$$p(z_1, \ldots, z_n) = \lim_{\epsilon \downarrow 0} \frac{p_\epsilon(z_1, \ldots, z_n)}{(\pi \epsilon^2)^n}$$

(8)

**Theorem:** (Hammersley)
Let $f$ be a Gaussian analytic function in a planar domain $D$, $z_1, \ldots, z_n \in D$, and consider the matrix $A = \left( \mathbf{E} f(z_i) f(z_j) \right)$. If $A$ is non-singular then $p(z_1, \ldots z_n)$ exists and equals

$$\mathbf{E} \left( |f'(z_1) \cdots f'(z_n)|^2 \left| \begin{array}{c} f(z_1) = \cdots = f(z_n) = 0 \end{array} \right. \right) \frac{1}{\det(\pi A)}.$$
**Theorem:** (Virág - P.)

The joint intensity of zeros for $f_U$ is

$$p(z_1, \ldots, z_n) = \pi^{-n} \det \left[ \frac{1}{(1 - z_i \bar{z}_j)^2} \right]_{i,j}$$

$$= \det[K(z_i, z_j)]$$

where $K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$ is the Bergman kernel for $U$. 
Proof of Determinantal Formula

Let
\[ T_\beta(z) = \frac{z - \beta}{1 - \overline{\beta}z} \]  \hspace{1cm} (9)
denote a Möbius transformation fixing the unit disk. Also, for fixed \( z_1, \ldots, z_n \in \mathbb{U} \) denote
\[ \Upsilon(z) = \prod_{j=1}^{n} T_{z_j}(z). \] \hspace{1cm} (10)

Key facts:

1. Let \( f = f_U \) and \( z_1, \ldots, z_n \in \mathbb{U} \). The distribution of the random function \( T_{z_1}(z) \cdots T_{z_n}(z)f(z) \) is the same as the conditional distribution of \( f(z) \) given \( f(z_1) = \ldots = f(z_n) = 0 \).
2. It follows that the conditional joint distribution of the random
variables \((f'(z_k) : k = 1, \ldots, n)\) given \(f(z_1) = \ldots = f(z_n) = 0,\)
is the same as the unconditional joint distribution of
\((\Upsilon'(z_k)f(z_k) : k = 1, \ldots, n)\).

3. Consider the \(n \times n\) matrices

\[
A_{jk} = \mathbb{E} f(z_j) f(z_k) = (1 - z_j \bar{z}_k)^{-1},
\]
\[
M_{jk} = (1 - z_j \bar{z}_k)^{-2}.
\]

By the classical Cauchy determinant formula,

\[
\text{det}(A) = \prod_{k, j} \frac{1}{1 - z_j \bar{z}_k} \prod_{k < j} (z_k - z_j)(\bar{z}_k - \bar{z}_j)
\]
\[
= \prod_{k=1}^{n} |\Upsilon'(z_k)|. \tag{11}
\]
4. We also use Borchardt’s identity:

\[ \text{perm} \left( \frac{1}{x_j + y_k} \right)_{j,k} \text{ det} \left( \frac{1}{x_j + y_k} \right)_{j,k} = \text{det} \left( \frac{1}{(x_j + y_k)^2} \right)_{j,k} \]

setting \( x_j = z_j^{-1} \) and \( y_k = -\overline{z_k} \) and dividing both sides by \( \prod_j z_j^2 \), gives that

\[ \text{perm}(A) \text{ det}(A) = \text{det}(M). \]  \hspace{1cm} (12)

5. Finally, recall the Gaussian moment formula: If \( Z_1, \ldots, Z_n \) are jointly complex Gaussian random variables with covariance matrix \( C_{j,k} = \mathbf{E} Z_j \overline{Z_k} \), then \( \mathbf{E}(|Z_1 \cdots Z_n|^2) = \text{perm}(C) \).
From Hammersley’s formula \( p(z_1, \ldots, z_n) \) equals
\[
\frac{E\left(|f'(z_1) \cdots f'(z_n)|^2 \right| f(z_1), \ldots, f(z_n) = 0)}{\pi^n \det(A)}.
\]
The numerator equals
\[
E\left(|f(z_1) \cdots f(z_n)|^2 \right) \prod_k |\Upsilon'(z_k)|^2 = \perm(A) \det(A)^2,
\]
where the last equality uses the Gaussian moment formula. Thus,
\[
p(z_1, \ldots, z_n) = \pi^{-n} \perm(A) \det(A)
= \pi^{-n} \det(M).
\]
**Theorem 2:** (Virág - P.)

Let

\[
X_k \sim \begin{cases} 
1 & r^{2k} \\
0 & 1 - r^{2k}
\end{cases}
\]

be independent. Then \(\sum_1^\infty X_k\) and \(N_r = |Z_U \cap B(0, r)|\) have same distribution.

**Corollary:** Let \(h_r = 4\pi r^2 / (1 - r^2)\) (hyperbolic area). Then

\[
P(N_r = 0) = e^{-h_r \frac{\pi}{24} + o(h_r)} = e^{-\frac{\pi^2/12 + o(1)}{1-r}}. \tag{13}
\]

All of the above generalize to other simply connected domains with smooth boundary.

\[
E \left( f_D(z) \overline{f_D(w)} \right) = 2\pi S_D(z, w) \quad \text{(Szegő Kernel)} \tag{14}
\]
Denote $q = r^2$. Key to law of $N_r = |\mathbf{Z}_U \cap B(0, r)|$:

$$
\mathbf{E}\left(\binom{N_r}{k}\right) = \frac{1}{k!} \int_{B_r^k} p(z_1, \ldots, z_k) dz_1, \ldots dz_k
$$

$$
= \frac{q^{\binom{k+1}{2}}}{(1 - q)(1 - q^2) \ldots (1 - q^k)}
$$

$$
= \gamma_k.
$$

Euler’s partition identity

$$
\sum_{k=0}^{\infty} \gamma_k s^k = \prod (1 + q^k s),
$$

implies that

$$
\mathbf{E}(1 + s)^{N_r} = \sum_{k=0}^{\infty} \mathbf{E}\left(\binom{N_r}{k}\right) s^k = \sum \gamma_k s^k
$$

has product form!
Dynamics

Let

\[ f_U(t, z) = \sum_{n} a_n(t) z^n \]  

(17)

with \( a_n(t) \) performing Ornstein-Uhlenbeck diffusion,
\( a_n(t) = e^{-t/2} W_n(e^t) \). Suppose that the zero set of \( f_U \) contains the origin. Movement of this zero satisfies stochastic differential equation

\[ dz = \sigma dW \]  

(18)

where

\[ \frac{1}{\sigma} = |f'_U(0)| = c \lim_{r \uparrow 1} \frac{1}{\sqrt{1 - r^2}} \prod_{z \in Z_U \atop 0 < |z| < r} |z| = \tilde{c} \prod_{k=1}^{\infty} e^{1/k} |z_k|. \]  

(19)