MATH H104 LECTURE XX, NOVEMBER 29, 2005

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Read compelet proof of taylor approximation in text. Recap, points reached:
$R_n(h) \;=\; f(x+h) - P_n(h)$
$P_n(h) \;=\; \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} \cdot h^k$

If $f \in C^{n+1}(a,b)$ and $x \in (a,\ b)$ then there exists θ between x and $x+h$ with $R_n(h) = \frac{f^{(n+1)}(\theta)}{(n+1)!} \cdot h^{n+1}$.

Proof. Consider $g(t) = 0$ at the origin and at $t = h$.
$g(t) = R_n(t) - \frac{t^{n+1}}{h^{n+1}} \cdot R_n(h)$
The kth derivative $g^{(k)}(t) = R_n^{(k)}(t) - (n+1)\cdot n \ldots (n-k+2)\frac{t^{n-k+1}}{h^{n+1}} \cdot R_n(h)$. Notice $g^{(i)}(0) = 0$ when i is in between 0 and n. Apply Rao's theorem, a special case of mean value theorem and g' will vanish at θ_1. Apply the theorem again and g'' vanishes at θ_2. Continue this and get $g^{(n+1)}(\theta_{n+1}) = 0$
$0 = R_n^{(n+1)}(\theta_{n+1}) - \frac{(n+1)!}{h^{n+1}} \cdot R_n(h)$
$f^{(n+1)}(x + \theta_{n+1}) = R_n^{(n+1)}(\theta_{n+1})$
$\theta = x + \theta_{n+1}$ □

Example: xpand $f(x) = e^x$ around $x = 0$.
$f(h) = \sum_{k=0}^{n} \frac{h^k}{k!} + R_n(h)$; $R_n(h) = \frac{e^\theta}{(n+1)!} \cdot h^{n+1}$
The taylor series is not guraranteed to always converge. Example:

$$f(x) = \begin{cases} e^{-x^{-2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$f^{(n)}(0) = 0$, thus the series doesn't converge. C^∞ is the set of always differentiable functions. For a real analytic functions, every point has a neighborhood where the taylor series converges. $f(x)$ is in C^∞ but it is not analytic.

Theorem 0.1. ***Weierstrass M-test:*** *suppose* $f_n : [a,\ b] \to \mathbb{R}$ *satisfies* $||f_n||_\infty \leq M_n$ *and* $\sum_n M_n < \infty$ *then* $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly in* $[a,\ b]$. *(*$\sum_1^N \delta_n(x)$ *converges uniformly as* $N \to \infty$*)*

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Proof. $F_n = \sum_{n=1}^{N} f_n$ is a Cauchy sequence in $C[a,\ b]$ because $\sum_{n=1}^{N} M_n$ is Cauchy. Given ϵ there exits N_0 such that for $N_1,\ N_2 \geq N_0,\ |\sum_{N_1+1}^{N_2} M_j| < \epsilon$.
$|F_{N_2}(x) - F_{N_1}(x)| = |\sum_{N_1+1}^{N_2} f_j(x)| < \epsilon$ $\square$

Riemann Integral. Given $f : [a,\ b] \to \mathbb{R}$ bounded $a = x_0 < x_1 < x_2 \ldots < x_n = b$ and $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots n$. The Riemann sum is defined as $\sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1})$. Call this partition pair $\tilde{P}$ and let $\Delta(\tilde{P}) = max_{1 \leq i \leq n}\ (x_i - x_{i-1})$ be its mesh. Riemann defined $\int_a^b f(x)dx$ to be $\lim_{\Delta(\tilde{P}) \to 0} R(f,\ \tilde{P})$. This limit equals L if for all $\epsilon > 0$ there exists $delta > 0$ such that $\Delta(\tilde{P}) < \delta$ then $|R(f,\ \tilde{P} - L| < \epsilon$. f is Riemann integrable in $[a,\ b]$ if this limit exists.

Theorem 0.2. *Any continuous* $f : [a,\ b] \to \mathbb{R}$ *is Riemann integrable.*

Darboux's idea: consider just one parition (no t_i) $a = x_0 < x_1 \ldots < x_n = b$. $m_i = inf_{[x_{i-1},\ x_i]}\ f$ and $M_i = sup_{[x_{i-1},\ x_i]}\ f$. Lower sum $L(f,\ P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$ and upper sum $U(f,\ P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$. Call f Darboux integrable if $sup_P L(f,\ P) = inf_P U(f,\ P)$ and in this case define the common value to be $\int_a^b f(x)dx$.

Theorem 0.3. f *is Darboux integrable if and only if* f *is Riemman integrable with the same value.*

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