Convergence of Series. Suppose $\alpha > 0$. Then
\[
\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty \quad \text{for } \alpha > 1
\]
\[
\infty \quad \text{for } \alpha \leq 1
\]

Blocking Test. Suppose $\{a_n\}_{n \geq 1}$ is a sequence which is decreasing (weakly) and $a_n > 0, \forall n$. Then
\[
\sum_{n=1}^{\infty} a_n < \infty \iff \sum_{m=1}^{\infty} 2^m a_{2^m} < \infty.
\]

Proof. Consider $A_m = \sum_{j=2^m}^{2^{m+1}-1} a_j \geq 2^m a_{2^m+1}$. However, we also have $A_m \leq 2^m a_{2^m}$. Let
\[
S_M = \sum_{m=1}^{M} 2^m a_{2^m} \geq \sum_{j=2}^{2^{M+1}-1} a_j \geq \frac{1}{2} \sum_{m=1}^{M} 2^{m+1} a_{2^m+1} = \frac{1}{2} (S_{M+1} - 2a_2).
\]

Now let $M \to \infty$. Note that if $2^M - 1 \leq N \leq 2^{M+1} - 1$, then $\sum_{j=1}^{2^{M-1}} a_j \leq \sum_{j=1}^{N} a_j \leq \sum_{j=1}^{2^{M+1}} a_j$. \qed

Example 0.1. For $\alpha > 0$,
\[
\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \iff \sum_{m=1}^{\infty} 2^m a_{2^m} = \sum_{m=1}^{\infty} 2^{m(1-\alpha)} < \infty
\]
because if $\alpha > 1$, the right-hand side above is a geometric series.

Example 0.2.
\[
\sum_{n=3}^{\infty} \frac{1}{n \log n}^\beta < \infty \iff \sum_{m=1}^{\infty} \frac{1}{(m \log 2)^\beta} < \infty
\]
The above converge if $\beta > 1$.

Example 0.3.
\[
\sum_{n=3}^{\infty} \frac{1}{n \log n (\log (\log n))^{\gamma}} < \infty \iff \sum_{m=1}^{\infty} \frac{1}{\log 2^m (\log (\log 2^m))^{\gamma}} = \sum_{m=1}^{\infty} \frac{1}{(\log 2)m (\log m + \log \log 2)^{\gamma}} < \infty
\]
The above converge if $\gamma > 1$. 

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Given \( a_n \in \mathbb{R} \), for which \( x \) does \( \sum_n a_n x^n \) converge? Two observations:

1. If \( \sum b_n \) converges, then \( b_n \to 0 \), because
   \[
   \sum_{n=1}^N b_n - \sum_{n=1}^{N-1} b_n = b_N.
   \]

2. Root test: If \( \sqrt[n]{|b_n|} \leq q < 1 \) for all \( n \geq n_0 \), then \( \sum b_n \) converges.

Note that boundedness of \( \sum_{n=1}^N b_n \) does not imply convergence. For example, let \( b_n = (-1)^n \). We need something more to show property (2) above. \( \mathbb{R} \) is complete, so we can show that the sequence of partial sums is Cauchy. Given an \( \epsilon > 0 \), we can find an \( n_0 \) such that for \( n > m > n_0 \),

\[
\left| \sum_{j=1}^{n} b_j - \sum_{j=1}^{m} b_j \right| = \sum_{j=m+1}^{n} |b_j| \leq \sum_{j=m+1}^{n} q^j < \epsilon.
\]

**Definition 0.4.** For \( b_n \in \mathbb{R} \), \( \limsup_{n \to \infty} b_n = \inf_{k \geq 1} \sup_{n \geq k} b_n \).

**Problem 0.5. (HW):** Show that \( \limsup_{n \to \infty} b_n = \sup_{n_j \to \infty} b_{n_j} \), where the sup is over all \( n_j \to \infty \) such that \( \lim b_{n_j} \) exists. Moreover, the sup is achieved and the equality is still true if both sides are \( +\infty \) or \( -\infty \).

**Problem 0.6. (HW):** Show that \( b_n \) converges if and only if \( \limsup_{n \to \infty} b_n = \liminf_{k \geq 1} \sup_{n \geq k} b_n \).

Consider the power series \( \sum a_n x^n \). If \( a_n x^n \to 0 \), then this series can’t converge. On the other hand, if \( |x||a_n|^{1/n} = \sqrt[n]{|a_n x^n|} \leq q < 1 \), then the series \( \sum a_n x^n \) does converge.

Observe that if \( a_n x^n \to 0 \), then \( \sqrt[n]{|a_n (xq)^n|} < q \) for all \( n \geq n_0 \), so \( \sum a_n (xq)^n \) converges, provided \( 0 < q < 1 \).

**Example 0.7.** Let \( \alpha \in \mathbb{R} \). Then \( \sum a_n x^n \) converges for \( |x| < 1 \), diverges for \( |x| > 1 \), and if \( |x| = 1 \), the answer can depend on \( \text{sign}(x) \). For example, \( \sum \frac{1}{n} \) diverges, but \( \sum (-1)^n \) converges. Note that \( \sqrt[n]{a} \to 1 \) as \( n \to \infty \).

**Ratio Test.** If \( b_n > 0 \) and \( \left| \frac{b_{n+1}}{b_n} \right| \leq q < 1 \) for all \( n \geq n_0 \), then \( \sum b_n < \infty \).

To complete the above example, for \( n \geq n_0 \), \( |b_n| \leq q^{n-n_0} |b_{n_0}| \), so the Cauchy condition applies. So \( \frac{(n+1)^n}{n^n} \to 1 \) and \( \frac{(n+1)^{n+1}}{n^n x^n} \to x \).

**Taylor Approximation.** Suppose \( f \) defined in \((a, b)\) is differentiable at \( x \). Write \( f(x+h) = f(x) + hf'(x) + R_1(h) \). (This defines \( R_1(h) \).) By definition, \( \frac{R_1(h)}{h} \to 0 \) as \( h \to 0 \). This can be expressed as \( R_1(h) = o(h) \) as \( h \to 0 \).

Suppose \( f \) is defined in \((a, b)\) and differentiable \( k - 1 \) times in \((a, b)\), and suppose also that \( f \) is differentiable \( k \) times at \( x \). Define \( P_k(h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \ldots + \frac{h^k}{k!} f^{(k)}(x) \) and \( R_k(h) = f(x+h) - P_k(h) \). Then \( R_k(h) = o(h^k) \) as \( h \to 0 \).

Moreover, if \( f \) is \( k + 1 \) times differentiable in \((a, b)\) then \( R_k(h) = \frac{f^{(k+1)}(\Theta)}{(k+1)!} h^{k+1} \) for some \( \Theta \) between \( x \) and \( x + h \).
Proof. \( P_k(0) = f(x), P_k'(0) = f'(x), \ldots, P_k^{(j)}(0) = f^{(j)}(x) \) for all \( j \leq k \).

Also, \( R_k^{(j)}(0) = 0 \) for \( j = 0, 1, \ldots, k \). By the Mean Value Theorem, \( R_k(h) = h R_k'(\Theta_1) \), since \( \frac{R_k(h) - R_k(0)}{h - 0} = R_k'(\Theta_1) \).

So we have
\[
R_k(h) = h R_1'(\Theta_1) = h \Theta_1 R_2''(\Theta_2) = h \Theta_1 \Theta_2 R_3'''(\Theta_3) = \ldots = h \Theta_1 \Theta_2 \ldots \Theta_{k-2} R_k^{(k-1)}(\Theta_{k-1}).
\]

Thus we have
\[
\frac{R_k(h)}{h^k} = \frac{h \Theta_1 \Theta_2 \ldots \Theta_{k-2} \Theta_{k-1} R_k^{(k-1)}(\Theta_{k-1})}{h^k},
\]
with the right-most term going to 0 as \( h \) goes to 0. \( \square \)