MATH H104 LECTURE 25, NOVEMBER 22, 2005

LECTURER: YUVAL PERES.

SCRIBE: JONATHAN GOLDMAN

Convergence of Series. Suppose $\alpha > 0$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{array}{ll} < \infty & \text{for } \alpha > 1 \\ = \infty & \text{for } \alpha \leq 1 \end{array} \tag{1}$$

Blocking Test. Suppose $\{a_n\}_{n\geq 1}$ is a sequence which is decreasing (weakly) and $a_n > 0, \forall n$. Then

$$\sum_{n=1}^{\infty} a_n < \infty \Longleftrightarrow \sum_{m=1}^{\infty} 2^m a_{2^m} < \infty.$$

Proof. Consider $A_m = \displaystyle\sum_{j=2^m}^{2^{m+1}-1} a_j \geq 2^m a_{2^{m+1}}$. However, we also have $A_m \leq 2^m a_{2^m}$. Let

$$S_M = \sum_{m=1}^{M} 2^m a_{2^m} \geq \sum_{j=2}^{2^{M+1}-1} a_j \geq \frac{1}{2}\sum_{m=1}^{M} 2^{m+1} a^{2^m+1} = \frac{1}{2}(S_{M+1} - 2a_2).$$

Now let $M \to \infty$. Note that if $2^M - 1 \leq N \leq 2^{M+1} - 1$, then $\displaystyle\sum_{j=1}^{2^M-1} a_j \leq \sum_{j=1}^{N} a_j \leq \sum_{j=1}^{2^{M+1}} a_j$. $\square$

Example 0.1. *For $\alpha > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \Longleftrightarrow \sum_{m=1}^{\infty} \frac{2^m}{2^{m\alpha}} = \sum_{m=1}^{\infty} 2^{m(1-\alpha)} < \infty$$

because if $\alpha > 1$, the right-hand side above is a geometric series.

Example 0.2.

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^\beta} < \infty \Longleftrightarrow \sum_{m=1}^{\infty} \frac{1}{(m \log 2)^\beta} < \infty$$

The above converge if $\beta > 1$.

Example 0.3.

$$\sum \frac{1}{n \log n (\log(\log n))^\gamma} < \infty \Longleftrightarrow \sum \frac{1}{\log 2^m (\log(\log 2^m))^\gamma} = \sum \frac{1}{(\log 2) m (\log m + \log\log 2)^\gamma} < \infty$$

The above converge if $\gamma > 1$.

Date: November 22, 2005.

Given $a_n \in \mathbb{R}$, for which x does $\sum_n a_n x^n$ converge? Two observations:

(1) If $\sum b_n$ converges, then $b_n \to 0$, because

$$\sum_{n=1}^{N} b_n - \sum_{n=1}^{N-1} b_n = b_N.$$

(2) Root test: If $\sqrt[n]{|b_n|} \le q < 1$ for all $n \ge n_0$, then $\sum b_n$ converges.

Note that boundedness of $\sum_{n=1}^{N} b_n$ does not imply convergence. For example, let $b_n = (-1)^n$. We need something more to show property (2) above. $\mathbb{R}$ is complete, so we can show that the sequence of partial sums is Cauchy. Given an $\epsilon > 0$, we can find an n_0 such that for $n > m > n_0$,

$$\left|\sum_{j=1}^{n} b_j - \sum_{j=1}^{m} b_j\right| = \sum_{j=m+1}^{n} |b_j| \le \sum_{j=m+1}^{n} q^j < \epsilon.$$

Definition 0.4. *For $b_n \in \mathbb{R}$, $\limsup_{n\to\infty} b_n = \inf_{k\ge 1} \sup_{n\ge k} b_n$.*

Problem 0.5. *(HW): Show that $\limsup b_n = \sup_{n_j} \lim_{j\to\infty} b_{n_j}$, where the $\sup$ is over all $n_j \to \infty$ such that $\lim b_{n_j}$ exists. Moreover, the $\sup$ is achieved and the equality is still true if both sides are $+\infty$ or $-\infty$.*

Problem 0.6. *(HW): Show that b_n converges if and only if $\limsup b_n = \liminf b_n$, where $\liminf b_n = \sup_{k\ge 1} \inf_{n\ge k} b_n$.*

Consider the power series $\sum a_n x^n$. If $a_n x^n \not\to 0$, then this series can't converge. On the other hand, if $|x||a_n|^{\frac{1}{n}} = \sqrt[n]{|a_n x^n|} \le q < 1$, then the series $\sum a_n x^n$ does converge.

Observe that if $a_n x^n \to 0$, then $\sqrt[n]{|a_n (xq)^n|} < q$ for all $n \ge n_0$, so $\sum a_n (xq)^n$ converges, provided $0 < q < 1$.

Example 0.7. *Let $\alpha \in \mathbb{R}$. Then $\sum n^\alpha x^n$ converges for $|x| < 1$, diverges for $|x| > 1$, and if $|x| = 1$, the answer can depend on sign(x). For example, $\sum \frac{1}{n}$ diverges, but $\sum \frac{(-1)^n}{n}$ converges. Note that $\sqrt[n]{n^\alpha} \to 1$ as $n \to \infty$.*

Ratio Test. If $b_n > 0$ and $\left|\frac{b_{n+1}}{b_n}\right| \le q < 1$ for all $n \ge n_0$, then $\sum b_n < \infty$.

To complete the above example, for $n \ge n_0$, $|b_n| \le q^{n-n_0}|b_{n_0}|$, so the Cauchy condition applies. So $\frac{(n+1)^\alpha}{n^\alpha} \to 1$ and $\dfrac{(n+1)^\alpha x^{n+1}}{n^\alpha x^n} \to x$.

Taylor Approximation. Suppose f defined in (a,b) is differentiable at x. Write $f(x+h) = f(x) + hf'(x) + R_1(h)$. (This defines $R_1(h)$.) By definition, $\frac{R_1(h)}{h} \to 0$ as $h \to 0$. This can be expressed as $R_1(h) = o(h)$ as $h \to 0$.

Suppose f is defined in (a,b) and differentiable $k-1$ times in (a,b), and suppose also that f is differentiable k times at x. Define $P_k(h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \ldots + \frac{f^{(k)}(x)}{k!}h^k$ and $R_k(h) = f(x+h) - P_k(h)$. Then $R_k(h) = o(h^k)$ as $h \to 0$.

Moreover, if f is $k+1$ times differentiable in (a,b) then $R_k(h) = \frac{f^{(k+1)}(\Theta)}{(k+1)!}h^{k+1}$ for some Θ between x and $x+h$.

Proof. $P_k(0) = f(x)$, $P_k'(0) = f'(x)$, ..., $P_k^{(j)}(0) = f^{(j)}(x)$ for all $j \leq k$.

Also, $R_k^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k$. By the Mean Value Theorem, $R_k(h) = hR_k'(\Theta_1)$, since $\frac{R_k(h)-R_k(0)}{h-0} = R_k'(\Theta_1)$.

So we have

$$R_k(h) = hR_1'(\Theta_1) = h\Theta_1 R_k''(\Theta_2) = h\Theta_1\Theta_2 R_k'''(\Theta_3) = \ldots = h\Theta_1\Theta_2\ldots\Theta_{k-2}R_k^{(k-1)}(\Theta_{k-1}).$$

Thus we have

$$\frac{R_k(h)}{h^k} = \frac{h\Theta_1\Theta_2\ldots\Theta_{k-2}\Theta_{k-1}}{h^k}\frac{R_k^{(k-1)}(\Theta_{k-1})}{\Theta_{k-1}},$$

with the right-most term going to 0 as h goes to 0. □

University of California, Berkeley
E-mail address: peres@stat.berkeley.edu