MATH H104 LECTURE, NOVEMBER 2, 2005

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Theorem 0.1. *Let X, Y be metric spaces. If $f_n : X \to Y$ are continuous and f_n converges to $f : X \to Y$ uniformly then f is continuous.*

Definition 0.2. *(Reminder - Uniform Convergent) $f_n \to f$ means that $d(f_n(x), f(x)) = ||f_n - f||_\infty \to^{n\to\infty} 0$ I.e. for all $\epsilon > 0$ there exists n_ϵ such that for all $n \geq n_\epsilon$ for all $x \in X$, $d(f_n(x), f(x) < \epsilon$.*

Example 0.3. *(Non-uniform convergence) Consider $f_n[0,1] \to [0,1]$; $f_n(x) = x^n$ $\lim_{n\to\infty} f(x) = \{0, x \neq 1; 1, x = 1\}$*

Uniformity fails. Take $\epsilon = 1/2$. There exists $n_x = n_x(x, \epsilon)$ for all $n \geq n_x(x, \epsilon)$, $x^n \leq \epsilon$.

Example 0.4. *(Another example) $f_n(x)$ is made up of the line from (0,0) to (1/n,1) to (2/n,0) to (1,0). Converges to the line from (0,0) to (1,0) but the distance $||f_n - f||_\infty = 1$.*

Proof. Given $\epsilon > 0$ and $x \in X$. Want $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. First find n such that $\sup_x d(f_n(x), f(x)) < \epsilon/3$. By continuity of f_n we know $\exists \delta > 0$ such that $f_n(B(x, \delta)) \subset B(f_n(x), \epsilon/3)$

To check $f(B(x, \delta)) \subset B(f(x), \epsilon)$ Let $z \in B(x, \delta)$. is $f(z) \in B(f(x), \epsilon)$? $d_y(f(z), f(x)) \leq d_y(f(z), f_n(z)) + d(f_n(z), f_n(x)) + d(f_n(x), f(x)) \leq \epsilon/3 + \epsilon/3 + \epsilon/3$. The first and last by choice of n, the middle by choice of δ.

In summary, first give an ϵ and then choose a sufficiently close f_n. □

Corollary 0.5. *If Y is a complete metric space $C(X, Y) = \{$ Continuous and bounded functions $X \to Y\}$ is complete for the metric $d_\infty(f, g) = \sup(d_y(f(x), g(x)))$. If $Y = \mathbb{R}$ this is $||f - g||_\infty$*

Reminder. $f : X \to Y$ is bounded if $f(x)$ is a bounded set, i.e. contained in some ball $B(y_0, R)$ with real, positive radius.

Note: We will use this only for $Y = \mathbb{R}$.

Proof. Given $\{f_n\}$ a Cauchy sequence in $C(X, Y)$ for each $x \in X$, $\{f_n(x)\}_{n=1}^\infty$ is Cauchy in Y. So it converges. Call the limit $f(x)$.

Need to show that f_n converges to f uniformly. That is, $d_\infty(f_n, f) \to 0$. Then it will follow that f itself is continuous and bounded as the uniform limit of bounded functions.

Given $\epsilon > 0$ there exists N such that for all $n, m > N$ we have $d_\infty(f_n, f_m) < \epsilon/2$. Claim that $d_\infty(f_n, f) < \epsilon$ for all $n \geq N$. Why? Fix $x \in X$. We know that there exists $m = m(x)$ with

Date: November 17, 2005.

$d_y(f_m(x), f(x)) < \epsilon/4$. Using the triangle inequality, $d_y(f_n(x), f(x)) \leq d_y(f_n(x), f_m(x)) + d_y(f_m(x), f(x)) \leq \epsilon/2 + \epsilon/4 \leq 3\epsilon/4$. By theorem this shows that f is continuous.

Why bounded? There exists n such that $d_\infty(f_n, f) \leq 1$. $f_n(x) \subset B(y_0, R) \implies f(x) \subset B(y_0, R+1)$. □

Definition 0.6. *A countable Intersection of open sets is called a G_δ-set.*

Theorem 0.7. *Let $f : X \to Y$ be any function where X, Y are metric spaces then $k_f = \{x \in X$ such that f is continuous at $x\}$ is a G_δ-set.*

Problem 0.8. *a) is there $f : A \to \mathbb{R}$ with $k_f = \mathbb{R} \setminus \mathbb{Q}$?*
b) is there $f : A \to \mathbb{R}$ with $k_f = \mathbb{Q}$?

Solution 0.9. *a) Yes. Example, by Riemann. $f(x) = \{0$ if x is irrational, $1/q$ if x is rational, not zero, and expressed as p/q where p, q are non zero, relatively prime, integers, and q is positive, and 1 if $x = 0\}$.*

if $x \in \mathbb{Q}$ then $f(x) \neq 0$, but there exists some sequence in $\mathbb{Q}$ that converges to x, so $x \in k_f$.

if $x \notin \mathbb{Q}$ Given $\epsilon > 0$, fin δ such that $\delta < \min(x - p/q)$ for $\delta < 1/\epsilon$ and $\delta < |x|$. Then $f(B(x, \delta)) \subset B(0, \epsilon)$.

b) No. Suppose $k_f = \mathbb{Q}$ By theorem $\mathbb{R} = \bigcap_{n=1}^{\infty} V_n$ with V_n open. Since $\mathbb{Q} \subset V_n, V_n$ is dense. This is impossible. Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$ as $D_{2n-1} = \mathbb{R} \setminus \{r_n\}, D_{2n} = V_n$. Then $\bigcap_{n=1}^{\infty} D_k = \emptyset$ contradicting Baire's theorem.

Definition 0.10. *Oscillation of f at x, $\omega_f(x) = \inf_{\delta>0}(diam(f(B(x, \delta))))$.*

Proof. (Of theorem) $k_f = \{x \in X$ such that for all $m \in \mathbb{N}$ there exists $\delta > 0$ with $f(B(x, \delta 0)) \subset B(f(x), 1/m)\}$ by definition of continuity. Need to check for all ϵ. If true for $1/m$ then for all ϵ use m such that $1/m < \epsilon$.

Note that $x \in k_f \iff \omega_f(x) = 0$. (Prove for homework.)

$k_f = \bigcap_{n=1}^{\infty}\{x$ such that $\omega_f(x) \leq 1/m\}$. Call $V_m = \{x$ such that $\omega_f(x) \leq 1/m\}$. Each V_m is open. $x \in V_m$ implies $\omega_f(x) < 1/m$ implies that there exists $\delta > 0$ such that $\mathrm{diam}(f(B(x, \delta))) < 1/m$. But then $\omega_f(z) < 1/m$ for all $z \in B(x, \delta)$. Since for any such z there is some $\gamma > 0$ with $B(z, \gamma) \subset B(x, \delta)$. $\gamma = \delta - d(z, x)$. Thus V_m is open. *$diam(f(B(z, \gamma)) < 1/m$.* □

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