Theorem 0.1. Let $X,Y$ be metric spaces. If $f_n : X \to Y$ are continuous and $f_n$ converges to $f : X \to Y$ uniformly then $f$ is continuous.

Definition 0.2. (Reminder - Uniform Convergent) $f_n \to f$ means that $d(f_n(x), f(x)) = \|f_n - f\|_\infty \to 0$ i.e. for all $\epsilon > 0$ there exists $n_\epsilon$ such that for all $n \geq n_\epsilon$ for all $x \in X$, $d(f_n(x), f(x)) < \epsilon$.

Example 0.3. (Non-uniform convergence) Consider $f_n[0,1] \to [0,1]; f_n(x) = x^n \lim_{n \to \infty} f(x) = \{0, x \neq 1; 1, x = 1\}$

Uniformity fails. Take $\epsilon = 1/2$. There exists $n_x = n_x(x,\epsilon)$ for all $n \geq n_x(x,\epsilon), x^n \leq \epsilon$.

Example 0.4. (Another example) $f_n(x)$ is made up of the line from $(0,0)$ to $\{1/n,1\}$ to $(1,0).$ Converges to the line from $(0,0)$ to $(1,0)$ but the distance $\|f_n - f\|_\infty = 1.$

Proof. Given $\epsilon > 0$ and $x \in X$. Want $\delta > 0$ such that $f(B(x,\delta)) \subset B(f(x),\epsilon)$. First find $n$ such that $\sup_x d(f_n(x), f(x)) < \epsilon/3$. By continuity of $f_n$ we know $\exists \delta > 0$ such that $f_n(B(x,\delta)) \subset B(f_n(x),\epsilon/3)$

To check $f(B(x,\delta)) \subset B(f(x),\epsilon)$ Let $z \in B(x,\delta)$. Is $f(z) \in B(f(x),\epsilon)$? $d_g(f(z), f(x)) \leq d_g(f(z), f_n(z)) + d(f_n(z), f_n(x)) + d(f_n(x), f(x)) \leq \epsilon/3 + \epsilon/3 + \epsilon/3$. The first and last by choice of $n$, the middle by choice of $\delta$.

In summary, first give an $\epsilon$ and then choose a sufficiently close $f_n$.

Corollary 0.5. If $Y$ is a complete metric space $C(X,Y) = \{ \text{Continuous and bounded functions } X \to Y \}$ is complete for the metric $d_\infty(f,g) = \sup(f(x), g(x)))$. If $Y = \mathbb{R}$ this is $\|f - g\|_\infty$

Reminder. $f : X \to Y$ is bounded if $f(x)$ is a bounded set, i.e. contained in some ball $B(y_0, R)$ with real, positive radius.

Note: We will use this only for $Y = \mathbb{R}$.

Proof. Given $\{f_n\}$ a Cauchy sequence in $C(X,Y)$ for each $x \in X, \{f_n(x)\}_{n=1}^\infty$ is Cauchy in $Y$. So it converges. Call the limit $f(x)$.

Need to show that $f_n$ converges to $f$ uniformly. That is, $d_\infty(f_n, f) \to 0$. Then it will follow that $f$ itself is continuous and bounded as the uniform limit of bounded functions.

Given $\epsilon > 0$ there exists $N$ such that for all $n,m > N$ we have $d_\infty(f_n, f_m) < \epsilon/2$. Claim that $d_\infty(f_n, f) < \epsilon$ for all $n \geq N$. Why? Fix $x \in X$. We know that there exists $m = m(x)$ with
A countable intersection of open sets is called a *Gδ*-set.

**Definition 0.6.** A countable intersection of open sets is called a *Gδ*-set.

**Theorem 0.7.** Let $f : X \to Y$ be any function where $X, Y$ are metric spaces then $k_f = \{ x \in X \text{ such that } f \text{ is continuous at } x \}$ is a *Gδ*-set.

**Problem 0.8.** a) is there $f : A \to \mathbb{R}$ with $k_f = \mathbb{R} \setminus \mathbb{Q}$?

b) is there $f : A \to \mathbb{R}$ with $k_f = \mathbb{Q}$?

**Solution 0.9.** a) Yes. Example, by Riemann. $f(x) = \{0$ if $x$ is irrational, $1/q$ if $x$ is rational, not zero, and expressed as $p/q$ where $p, q$ are non zero, relatively prime, integers, and $q$ is positive, and $1$ if $x = 0$.

If $x \in \mathbb{Q}$ then $f(x) \neq 0$, but there exists some sequence in $\mathbb{Q}$ that converges to $x$, so $x \in k_f$.

If $x \notin \mathbb{Q}$ Given $\epsilon > 0$, fin $\delta$ such that $\delta < \min(x - p/q)$ for $\delta < 1/\epsilon$ and $\delta < |x|$. Then $f(B(x, \delta)) \subset B(0, \epsilon)$.

b) No. Suppose $k_f = \mathbb{Q}$ By theorem $\mathbb{R} = \bigcap_{n=1}^{\infty} V_n$ with $V_n$ open. Since $\mathbb{Q} \subset V_n, V_n$ is dense. This is impossible. Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$ as $D_{2n-1} = \mathbb{R} \setminus \{r_n\}, D_{2n} = V_n$. Then $\bigcap_{n=1}^{\infty} D_k = \emptyset$ contradicting Baire’s theorem.

**Definition 0.10.** Oscillation of $f$ at $x$, $\omega_f(x) = \inf_{\delta > 0}(\text{diam}(f(B(x, \delta))))$.

**Proof.** (Of theorem) $k_f = \{ x \in X \text{ such that for all } m \in \mathbb{N} \text{ there exists } \delta > 0 \text{ with } f(B(x, \delta)) \subset B(f(x), 1/m) \}$ by definition of continuity. Need to check for all $\epsilon$. If true for $1/m$ then for all $\epsilon$ use $m$ such that $1/m < \epsilon$.

Note that $x \in k_f \iff \omega_f(x) = 0$. (Prove for homework.)

$k_f = \bigcap_{n=1}^{\infty} \{ x \text{ such that } \omega_f(x) \leq 1/m \}$. Call $V_m = \{ x \text{ such that } \omega_f(x) \leq 1/m \}$. Each $V_m$ is open. $x \in V_m$ implies $\omega_f(x) < 1/m$ implies that there exists $\delta > 0$ such that $\text{diam}(f(B(x, \delta))) < 1/m$. But then $\omega_f(z) < 1/m$ for all $z \in B(x, \delta)$. Since for any such $z$ there is some $\gamma > 0$ with $B(z, \gamma) \subset B(x, \delta)$. $\gamma = \delta - d(z, x)$. Thus $V_m$ is open. $\text{diam}(f(B(z, \gamma))) < 1/m$. □

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