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Theorem 0.1 (Darboux's Theorem). *Given f differentiable on $[a,b]$, defined in a neighborhood of $[a,b]$, f' attains any value between $f'(a)$ and $f'(b)$.*

Proof. Suppose $f'(a) < f'(b)$. Let $L \in (f'(a), f'(b))$. For small h, consider $S_h(x) = \frac{f(x+h)-f(x)}{h}$ where $x \in [a, b-h]$.

$$\left.\begin{array}{rcl} \lim_{h\downarrow 0} S_h(a) & = & f'(a) \\ \lim_{h\downarrow 0} S_h(b-h) & = & f'(b) \end{array}\right\} \tag{1}$$

For h small enough, we will have both $S_h(a) < L$ and $S_h(b-a) > L$. Fix an h that satisfies this.

f is continuous on $[a,b]$, so S_n is continuous on $[a, b-h]$. By Intermediate Value Theorem, $\exists c \in [a, b-h]$ with $S_h(c) = L$. By Mean Value Theorem, $\exists \theta \in (c, c+h)$ with $f'(\theta) = L$.

For the case where $f'(a) > f'(b)$, just mimic the above, or apply the same argument to $-f$. □

Theorem 0.2 (Ratio Mean Value Theorem). *Suppose $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b), and $g : [a,b] \to \mathbb{R}$ continous and differentiable on (a,b). Assume that $g(a) \neq g(b)$ and $f(a) \neq f(b)$. Then $\exists \theta \in (a,b)$ with $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\theta)}{g'(\theta)}$. (Special case: $g(x) = x$ is Mean Value Theorem.)*

Proof. Let $h(x) = g(x)[f(b) - f(a)] - f(x)[g(b) - g(a)]$.

$$\begin{array}{rcl} h(b) - h(a) & = & (g(b) - g(a))[f(b) - f(a)] - (f(b) - f(a))[g(b) - g(a)] \\ h(b) - h(a) & = & 0 \end{array}$$

So by the regular Mean Value Theorem, $\exists \theta$, such that $h'(\theta) = 0$, so

$$h'(\theta) = 0 = g'(\theta)[f(b) - f(a)] - f'(\theta)[g(b) - g(a)]$$

□

Remark 0.3. *In book, apply this theorem to prove L'Hospital's rule.*

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