

## MATH H104 LECTURE 20, NOVEMBER 10, 2005

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**Definition 20.1:** A metric space  $X$  is separable if there is a countable set  $D \subset X$  which is dense in  $X$ . (Examples:  $\mathbb{R}$  is separable.  $\mathbb{R}^n$  is separable;  $\mathbb{Q}^n \subset \mathbb{R}^n$  is dense.) More generally, suppose  $(X, d)$  is separable, and so is  $(Y, \rho)$ . Then  $(X * Y, d_p)$  is a metric space for  $1 \leq p \leq \infty$ . We will focus on  $p = 1, 2, \infty$ .

(Note:  $d_p((x_1, y_1), (x_2, y_2)) = [d(x_1, x_2)^p + \rho(y_1, y_2)^p]^{1/p}$ ; for  $p = \infty$ ,  $d_\infty((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), \rho(y_1, y_2))$ .)

Claim:  $(X * Y, d_p)$  is separable.  $d_1 \geq d_\infty$ . This suggests that  $d_1 \geq d_p \forall p$ . In fact,  $d_p$  is monotone decreasing according to  $p$ . (Exercise: Show that for  $a, b \geq 0$ ,  $(a^p + b^p)^{1/p}$  is decreasing in  $p$ .)

Let  $D_X$  be dense in  $X$  and  $D_Y$  be dense in  $Y$ .  $D_X \subset X, D_Y \subset Y \Rightarrow D_X \times D_Y \subset X \times Y$ . For any  $\epsilon > 0$ ,  $x \in X \Rightarrow B_d(x, \epsilon) \cap D_X \neq \emptyset$ , and  $y \in Y \Rightarrow B_\rho(y, \epsilon) \cap D_Y \neq \emptyset$ .  $B_d(x, \frac{\epsilon}{2}) \times B_\rho(y, \frac{\epsilon}{2}) \subset B_1((x, y), \epsilon) \subset B_p((x, y), \epsilon)$  (A  $B_p$  ball uses the  $d_p$  metric), since  $d_1((x_1, y_1), (x_2, y_2)) < \epsilon$ . Thus,  $\forall \epsilon > 0, (x, y) \in X \times Y, B_p((x, y), \epsilon) \neq \emptyset$ .

**Theorem 20.2 (Weierstrass Approximation Theorem):** The set of polynomials is dense in  $C[a, b]$  with the max norm for  $a < b$  in  $\mathbb{R}$ . This result leads quickly to separability of  $C[a, b]$ .

Given  $f \in C[a, b]$ , find a polynomial  $P$  with  $\|f - p\|_\infty < \frac{\epsilon}{2}$ . Now approximate each coefficient  $a_i$  in the polynomial by a rational  $q_i$  satisfying  $|q_i - a_i| < \frac{\epsilon}{2(n+1)b^n}$ . Then for all  $t \in [a, b]$ ,  $|q_i^{t^i} - a_i^{t^i}| < \frac{\epsilon}{2(n+1)} \Rightarrow p' = \sum_{i=0}^n q_i t^i \Rightarrow |p'(t) - p(t)| < \frac{\epsilon}{2}$ .  $p'$  is a polynomial with rational coefficients.  $\mathbb{Q}[t]$  is countable, since it is a countable union of countable sets. Important: the powers must not be infinite.

**Theorem 20.3:** Any compact metric space is separable. More generally, a totally bounded metric space  $X$  is separable.

$\forall \epsilon = \frac{1}{k}, \exists x_1^{(k)}, \dots, x_{N_k}^{(k)}$  with  $X = \bigcup_{i=1}^{N_k} B(x_i^{(k)}, \frac{1}{k}) \Rightarrow \bigcup_{k=1}^{\infty} \{x_i^{(k)}\}_{i=1}^{N_k}$  is dense.

Suppose  $X$  is bounded and separable. Is  $X$  totally bounded? No, take  $X$  a countable and discrete space.

**Theorem 20.4:** Suppose  $(X, d)$  is separable and  $Y \subset X$ . Then  $(Y, d)$  is separable. Let  $E \subset X$  be countable and dense in  $X$ .  $\forall k, \bigcap_{x \in E} B_x(x, \frac{1}{k}) = X \supset Y$ . For  $x \in E$  such that

$B_x(x, \frac{1}{k})$  intersects  $Y$ , pick  $y_k(x) \in Y$  in this ball. Write  $A_k = \{y_k^{(x)}\}_{x \in E}$  and  $A = \bigcap_{k=1}^{\infty} A_k$ . Then  $A$  is countable and dense in  $Y$ .

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