Problem Set 1

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September 7, 2005

Problem 9

(a) Clearly $B + B'$ is disjoint from $A + A'$, because otherwise we'd have an element x in the intersection of the two sets. We'd have $x = a + a' = b + b'$ for some $a \in A, a' \in A', b \in B, b' \in B'$. But $a < b$ and $a' < b'$ so it is always the case that $a + a' < b + b'$, so it's impossible for $a + a' = b + b'$. Now let $x = A|B$ be the cut representing $-\sqrt{2}$ and $x' = A'|B'$ be the cut representing $\sqrt{2}$. More precisely, $A = \{a \in \mathbb{Q} : a < 0$ and $a^2 > 2\}$, $B = \{b \in \mathbb{Q} : b > 0$ or $b^2 < 2\}$. $A' = \{a' \in \mathbb{Q} : a' < 0$ or $a'^2 < 2\}$, $B' = \{b' \in \mathbb{Q} : b' > 0$ and $b'^2 > 2\}$. We show that neither $A + A'$ nor $B + B'$ contain 0.

First consider $A + A'$. There are two cases. Suppose $a \in A, a' \in A', a' < 0$. Since $a < 0$ and $a' < 0$ we must have $a + a' < 0$. Now suppose $a' \geq 0$ and $a'^2 < 2$. We must $a^2 > a'^2 \Rightarrow a^2 - a'^2 > 0 \Rightarrow (a + a')(a - a') > 0$. Since $a < 0, (a - a')$ is negative, and so $(a + a')$ must be negative for the product to be positive. Thus we have that $a + a' < 0$. Therefore $0 \notin A$.

Now consider $B + B'$. There are similarly two cases. Suppose $b \in B, b' \in B', b > 0$. Then we have $b > 0$ and $b' > 0$ and so $b + b' > 0$. Now suppose $b > 0$ and $b^2 < 2$. We have $b'^2 > b^2 \Rightarrow b'^2 - b^2 > 0 \Rightarrow (b' + b)(b' - b) > 0$. Thus we must have $(b' + b) \neq 0$. Therefore, 0 is not in $B + B'$ either.

Therefore 0 is not in the union of $A + A'$ and $B + B'$, and so the union does not form all of $\mathbb{Q}$.

(b) Take x and x' as in part (a). We showed above that $(A + A') \cup (B + B')$ was not all of $\mathbb{Q}$. Thus the sum $x + x' = (A + A')|(B + B')$ is not a cut, because by definition $(A + A') \cup (B + B')$ must form all of $\mathbb{Q}$. Since the sum of two cuts is not a cut in this case, $\mathbb{R}$ is not closed under addition under this definition, and so this must be an incorrect definition of addition.

(c) Let $x = x' = A|B$ be the cut representing 0. Here A is the set of rationals < 0. Consider $x \cdot x' = (A \cdot A)|\text{rest of } \mathbb{Q}$. $A \cdot A = \{r : \exists a, a' \in A, r = a \cdot a'\}$. Clearly $A \cdot A$ consists of only positive rationals. Therefore, the "rest of $\mathbb{Q}$" contains negative rationals, which violates the condition that for a cut $C|D, \forall c \in C, \forall d \in D, c < d$. Therefore we cannot define multiplication as $x \cdot x' = (A \cdot A')|\text{rest of } \mathbb{Q}$.

Problem 15

Recall that we are given $x > 0$ and $n \in \mathbb{N}$. We claim that $y = \text{l.u.b.}(S)$, where $S = \{s \in \mathbb{R} : s^n \leq x\}$. First we show that S is both nonempty and bounded above, and then we show that y^n can be neither greater than or less than x, and then that y is unique.

Clearly, $0^n \leq x$, so $0 \in S$ and thus S is nonempty. Now consider the number $z = (\frac{x}{n} + 1)$. We claim that z is an upper bound for S. Suppose there is a number u that is greater than z. Then

$$u^n > z^n = \left(\frac{x}{n} + 1\right)^n = \sum_{i=1}^{n} \binom{n}{i} \left(\frac{x}{n}\right)^{n-i}$$

by the binomial theorem. Now we simply extract one term from that summation and get

$$u^n > \sum_{i=1}^{n} \binom{n}{i} \left(\frac{x}{n}\right)^{n-i} > \binom{n}{n-1} \left(\frac{x}{n}\right)^{n-(n-1)} = n\frac{x}{n} = x.$$

Thus any number u which is greater than z is greater than x when taken to the nth power, and therefore u is not in the set S. Therefore z forms an upper bound for S. Since S is both nonempty and bounded above, by the completeness of $\mathbb{R}$ it has a least upper bound, which we call y.

Now suppose $y^n > x$. Let $\epsilon = y^n - x$. We know from exercise 14 that there exists a number δ for which all u such that $|y - u| < \delta$ are such that $|y^n - u^n| < \epsilon$. Let $h > 0$ be less than δ. Then we have $|y - (y - h)| < \delta$ and therefore that $|y^n - (y - h)^n| < \epsilon$. Therefore $(y - h)^n > y^n - \epsilon = x$ and so $(y - h)^n > x$. But $y - h$ is both less than y and greater than x when taken to the nth power. Therefore it is an upper bound for S that is less y, contradicting the fact that y was the least upper bound for S. Therefore we cannot have $y^n > x$.

Now suppose $y^n < x$. Let $\epsilon = x - y^n$. From exercise 14, we know that there exists a number δ for which all u such that $|y - u| < \delta$ are such that $|y^n - u^n| < \epsilon$. Let h be less than δ. Then $|(y + h) - y| < \delta$ and therefore $|(y + h)^n - y^n| < \epsilon$. This means that $(y + h)^n < y^n + \epsilon = x$ and so $(y + h)^n < x$. Thus $y + h$ is a number greater than y for which $(y + h)^n < x$. Thus y is not actually an upper bound for S, contradicting our assumption. So we cannot have $y^n < x$.

So we know we must have $x = y^n$. Now we show that y is unique. Suppose that $z^n = x$.

Then we have

$$0 = z^n - y^n = (z-y)\sum_{i=1}^{n}(z^{n-1-i}y^i).$$

This implies that one of the two factors above is 0. But the factor on the right, the summation factor, is greater than 0, since both z and y are positive. Therefore, the left factor is 0, and we have $(z-y)=0$ and therefore that $z=y$.

Problem 16

(a) Observe the following facts which follow from the definition of x_k. In general, x_k is the largest integer less than

$$10^k\left(x-\left(N+\sum_{i=1}^{k-1}\frac{x_i}{10^i}\right)\right).$$

Thus we have that

$$10^k\left(x-\left(N+\sum_{i=1}^{k-1}\frac{x_i}{10^i}\right)\right)-x_k<1.$$

The proof is by induction on k.
Base case: $x-N<1$, since otherwise $N+1$ would be an integer less than x. And clearly $x-N\geq 0$.Thus $0\leq 10(x-N)<10$. Thus x_1, the largest integer less than $10(x-N)$, must be a digit between 0 and 9.
Now assume that x_{m-1} is a digit between 0 and 9, and therefore that

$$10^{m-1}\left(x-\left(N+\sum_{i=1}^{m-2}\frac{x_i}{10^i}\right)\right)-x_{m-1}<1.$$

By definition, x_m is the largest integer less than

$$10^m\left(x-\left(N+\sum_{i=1}^{m-1}\frac{x_i}{10^i}\right)\right)=10\left(10^{m-1}\left(x-\left(N+\sum_{i=1}^{m-2}\frac{x_i}{10^i}\right)-\frac{x_{m-1}}{10^{m-1}}\right)\right)=$$

$$10\left(\left[10^{m-1}\left(x-\left(N+\sum_{i=1}^{m-2}\frac{x_i}{10^i}\right)\right)\right]-x_{m-1}\right).$$

We know from the definition of x_{m-1} that

$$\left[10^{m-1}\left(x-\left(N+\sum_{i=1}^{m-2}\frac{x_i}{10^i}\right)\right)\right]-x_{m-1}<1.$$

Thus

$$10\left(\left[10^{m-1}\left(x-\left(N+\sum_{i=1}^{m-2}\frac{x_i}{10^i}\right)\right)\right]-x_{m-1}\right)<10,$$

and so x_m is the largest integer less than 10, and so must be a digit between 0 and 9.

(b) We first consider the simpler case of 0.999... and then extend this to a general case of $N.x_1x_2...x_{k-1}999...$ Let's assume that the number $x = 0.999...$ can be produced with the given algorithm. We know that x is less than 1, because otherwise the digit before the decimal point would be a 1 and not a 0. We also observe that x is greater than any number terminating in a finite string of 9s, i.e. 0.9, 0.99, 0.999, etc. It is easy to see which numbers generated these strings, namely $\frac{9}{10} = 0.9, \frac{9}{10} + \frac{9}{100} = 0.99$, etc. Thus we must have

$$x > \sum_{i=1}^{n} \frac{9}{10^i}.$$

But this is true for any arbitrary n, so we must have

$$x \geq \sum_{i=1}^{\infty} \frac{9}{10^i} = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1$$

by the infinite geometric series sum formula. So we have

$$1 > x \geq \sum_{i=1}^{\infty} \frac{9}{10^i} = 1,$$

an obvious contradiction. Thus it is impossible to produce the number 0.999...

Now consider the general case of a number $x = N.x_1x_2...x_{k-1}999...$, where x_{k-1} is the last digit that is not a 9, and the related number $y = N.x_1x_2...x_{k-1}000....$ We can write x as $x = y + 0.00...0999...$ where the 9s start in the kth decimal place. We know that $\frac{1}{10^{k-1}} \geq 0.00...0999...$ because otherwise the digit x_{k-1} could be one greater. Thus we have

$$y + \frac{1}{10^{k-1}} > x = y + 0.00...999... \geq y + \sum_{i=1}^{\infty} \frac{9}{10^{k-1+i}} = y + \frac{1}{10^{k-1}} \sum_{i=1}^{\infty} \frac{9}{10^i} = y + \frac{1}{10^{k-1}},$$

the last step again using the infinite series sum formula. In short, we have

$$y + \frac{1}{10^{k-1}} > x \geq y + \frac{1}{10^{k-1}}$$

which is impossible. So we cannot generate a number x as above which terminates in an infinite string of 9s.

(c) If we begin counting at 0 for simplicity, then the kth element of the set $\{N, N + \frac{x_1}{10}, n + \frac{x_1}{10} + \frac{x_2}{100}, ...\}$, call S_k, is $N + \sum_{i=1}^{k} \frac{x_i}{10^i}$. Thus the decimal expansion of the kth element is $N.x_1x_2...x_k000....$ This is clearly less than or equal to $N.x_1x_2x_3...$ since each of the digits $\{x_{k+1}, x_{k+2}, ...\}$ are each ≥ 0. Thus the set is bounded, and $N.x_1x_2x_3...$ is an upper bound. So by Theorem 2, there exists a least upper bound for the set. Suppose $M.y_1y_2y_3...$ is the least upper bound for the set. Then for any k, we must have $M.y_1y_2y_3... - S_k \leq N.x_1x_2x_3... - S_k = 0.0...0x_{k+1}x_{k+2}....$ Thus M and N, and the first k digits of $M.y_1y_2y_3...$ and $N.x_1x_2x_3...$, must agree. But this is true for any k, and so we must have $M.y_1y_2y_3... = N.x_1x_2x_3....$ So $N.x_1x_2x_3...$ is the least upper bound for the set.

Extra Problem

Suppose $x \in \mathbb{Q}$ solves

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0,$$

with $a_i \in \mathbb{Z}$. Let $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}, q \neq 0$ and $gcd(p, q) = 1$. We rewrite the equation as

$$x^5 = -a_4x^4 - a_3x^3 - a_2x^2 - a_1x - a_0$$

and substitute for x, giving

$$\frac{p^5}{q^5} = -a_4\frac{p^4}{q^4} - a_3\frac{p^3}{q^3} - a_2\frac{p^2}{q^2} - a_1\frac{p}{q} - a0.$$

Multiplying through by q^5 gives us

$$p^5 = -a_4p^4q - a_3p^3q^2 - a_2p^2q^3 - a_1pq^4 - a_0q^5.$$

Since all of $ai, p, q \in \mathbb{Z}$, the right-hand side of this equation is an integer. Each term has a common factor of q, thus the right-hand side of the equation is an integer divisible by q, and therefore the left-hand side must also be an integer divisible by q. Thus $q|p^5$. But q and p are relatively prime, and so it must be the case that $q = 1$ or $q = -1$. So we have that $x = p$ or $x = -p$, and in either case $x \in \mathbb{Z}$.