Solutions to homework 13
Statistics 205B: Spring 2008

1. (Problem 4.1 from section 7.4 in Durrett)
   (a) Generalize the proof of 7.4.6 to conclude that if \( u < v \leq a \) then
   \[
P_0(T_a < t, u < B_t < v) = P_0(2a - v < B_t < 2a - u).
   \]
   (b) Let \( M_t = \max_{0 \leq s \leq t} B_s \). Use (a) to derive the joint density
   \[
P_0(M_t = a, B_t = x) = \frac{2(2a - x)}{\sqrt{2\pi t^3}} e^{-(2a-x)^2/2t}.
   \]
   **Solution:**
   (a) Let \( Y_s(w) = 1 \{ s < t, u < w(t-s) < v \} \) and \( \tilde{Y}_s(w) = 1 \{ s < t, 2a-v < w(t-s) < 2a-u \} \). We have \( E_n Y_s = E_n \tilde{Y}_s \). Now use Markov property for the stopping time \( S = \inf \{ s < t : B_s = a \} \).
   (b) Note that \( P_0(M_t > a, u < B_t < v) = P_0(2a - v < B_t < 2a - u) \).

   From this we have
   \[
P_0(M_t > a, B_t = x) = P_0(B_t = 2a - x) = (2\pi t)^{-1/2} e^{-(2a-x)^2/2t}.
   \]
   Now differentiate w.r.t. \( a \).

2. (Problem 6.2 from section 7.6 in Durrett)

   Suppose \( S_n \) is one-dimensional simple random walk and let
   \[
   R_n = 1 + \max_{m \leq n} S_m - \min_{m \leq n} S_m
   \]
   be the number of points visited by time \( n \). Show that \( R_n/\sqrt{n} \implies a \) limit.
   **Solution:** \( \phi(\omega) = \max_{0 \leq s \leq 1} \omega(s) - \min_{0 \leq s \leq 1} \omega(s) \) is continuous so we have
   \[
   \frac{1}{\sqrt{n}} (\max_{m \leq n} S_m - \min_{m \leq n} S_m) \implies \max_{0 \leq s \leq 1} B_s - \min_{0 \leq s \leq 1} B_s.
   \]

3. (Problem 6.3 from section 7.6 in Durrett)

   If \( X_1, X_2, \ldots \) are i.i.d. with \( E X_i = 0 \) and \( E X_i^2 = 1 \), then from example 7.6.5 we have
   \[
n^{-3/2} \sum_{m=1}^{n} (n + 1 - m) X_m \implies \int_0^1 B_t dt.
   \]
(a) Show that the right hand side has a normal distribution with mean 0 and variance 1/3.

(b) Deduce the result from the Lindeberg-Feller theorem.

Solution:

(a) Clearly \((1/n) \sum_{m=1}^{n} B(m/n)\) has a normal distribution. The sums converges a.s. and hence in distribuition to \(\int_0^1 B_t \, dt\), so by Exercise 3.9 the integral has a normal distribution. To compute the variance, we write

\[
\mathbb{E} \left( \int_0^1 B_t \, dt \right)^2 = \mathbb{E} \left( \int_0^1 \int_0^1 B_s B_t \, ds \, dt \right) = 2 \int_0^1 \int_s^1 \mathbb{E}(B_s B_t) \, ds \, dt = \frac{1}{3}.
\]

(b) Let \(X_{n,m} = (n + 1 - m)X_m/n^{3/2}\). \(\mathbb{E}X_{n,m} = 0\) and \(\sum_{m=1}^{n} \mathbb{E}X_{n,m}^2 = n^{-3} \sum_{j=1}^{n} j^2 \to 1/3\). Note that \(\mathbb{E}(X_{n,m}^2; |X_{n,m}| > \varepsilon) \leq n^{-1} \mathbb{E}(X_1^2; |X_1| > \varepsilon \sqrt{n})\). Hence the sum in (ii) in (4.5) in chapter 2 is \(\leq \mathbb{E}(X_1^2; |X_1| > \varepsilon \sqrt{n}) \to 0\) by dominated convergence.

4. (Problem 9.2 from section 7.9 in Durrett)

Show that if \(\mathbb{E}|X_i|^{\alpha} = \infty\) for some \(\alpha < 2\) then

\[
\lim_{n \to \infty} \sup |S_n|/n^{1/\alpha} = \infty \text{ a.s.}
\]

so the law of iterated logarithm fails.

**Hint:** First show that \(\lim_{n \to \infty} \sup |X_n|/n^{1/\alpha} = \infty \text{ a.s.}\)

**Solution:** \(\mathbb{E}|X_i|^{\alpha} = \infty\) implies \(\sum_{m=1}^{\infty} \mathbb{P}(|X_i| > Cn^{1/\alpha}) = \infty\) for any \(C > 0\). Using the second Borel-Cantelli now we see that \(\lim_{n \to \infty} \sup |X_n|/n^{1/\alpha} = \infty\). Since \(\max\{|S_n|, |S_{n-1}|\} \geq |X_n|/2\) it follows that \(\lim_{n \to \infty} \sup S_n/n^{1/\alpha} = \infty\).

5. (Problem 9.3 from section 7.9 in Durrett)

Give a direct proof that the limit set of \(\{S_n/(2n \log \log n)^{1/2}\}\) is \([-1, 1]\), where \(X_1, X_2, \ldots\) are i.i.d. with \(\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1\) and \(S_n = X_1 + \cdots + X_n\).

**Hint:** Use the law of iterated logarithm to get the extreme points and then fill the middle points.

**Solution:** LIL implies that

\[
\lim_{n \to \infty} \sup S_n/(2n \log \log n)^{1/2} = 1, \lim_{n \to \infty} \inf S_n/(2n \log \log n)^{1/2} = -1
\]

so the limit set is contained in \([-1, 1]\). On the other hand since \(\mathbb{E}X_i^2 = 1 < \infty\)

\[
\sum_{m=1}^{\infty} \mathbb{P}(X_n > \varepsilon \sqrt{n}) < \infty
\]
for any $\varepsilon > 0$. So $X_n/\sqrt{n} \to 0$. This shows that the differences $(S_{n+1} - S_n)/\sqrt{n} \to 0$. Let $Y_n = S_n/(2n \log \log n)^{1/2}$. Then we have

$$\limsup_{n \to \infty} |Y_n - Y_{n-1}| \leq \limsup_{n \to \infty} \left| \frac{X_n}{\sqrt{2n \log \log n}} \right| + \limsup_{n \to \infty} \left| Y_{n-1} - 1 \right| \left( 1 - \sqrt{\frac{2(n-1) \log \log (n-1)}{2n \log \log n}} \right) = 0.$$

So as $S_n/(2n \log \log n)^{1/2}$ wanders back and forth between $-1$ and $1$ it fills up the entire interval.