ASYMPTOTIC VARIANCE of the MLE

Maximum likelihood estimators typically have good properties when the sample size is large.

Suppose $X_1, \ldots, X_n$ are iid from some distribution F_{θ_o} with density f_{θ_o}. We observe data $x_1, \ldots, x_n$.

The Likelihood is:

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

and the log likelihood is:

$$l(\theta) = \sum_{i=1}^{n} log[f_\theta(x_i)]$$

The maximum likelhood estimator maximizes $l(\cdot)$, and is an estimator for θ_o. Under some regularity conditions of f_θ, for n large

$$\hat{\theta} \approx N(\theta_o, \frac{1}{nI(\theta_o)})$$

where $I(\theta)$ is called the *information*, and is defined as

$$I(\theta) = E(\frac{\partial log f_\theta(X)}{\partial \theta})^2$$

Notice that X is capitalized above. It denotes that the expectation is beinf taken with respect to X and its distribution. Also note that the derivative is with repect to θ.

It is often that case that the information also can be expressed as

$$I(\theta) = -E(\frac{\partial^2 log f_\theta(X)}{\partial \theta^2})$$

For $\tilde{\theta}$ any unbiased estmator for θ_o, we have a lower bound on the variance of $\tilde{\theta}$

$$Var(\tilde{\theta}) \geq \frac{1}{nI(\theta_o)}$$

Let's work some examples.

Example: $X_1, \ldots, X_n$ iid with density $f_\sigma(x) = \frac{1}{2\sigma} exp(-|x|/\sigma)$.

Method of Moments

1. $E(X) = 0$ by symmetry

$E(X^2) = \int_{-\infty}^{\infty} x^2 f_\sigma(x) dx =$

2. $\sigma =$

3. $\hat{\sigma} =$

Method of Moments

1. Likelihood function: $L(\sigma) =$

log likelihood $l(\sigma) =$

2. Differentiate with respect to σ

$$\frac{\partial logl(\sigma)}{\partial \sigma} =$$

3. Set to 0 and solve for $\tilde{\sigma}$

Asymptotic Variance of $\tilde{\sigma}$

$\frac{\partial log f_\sigma(x)}{\partial \sigma} =$

$\frac{\partial^2 log f_\sigma(x)}{\partial \sigma^2} =$

$I(\sigma) = -E[\frac{\partial^2 log f_\sigma(x)}{\partial \sigma^2}] =$

The variance of $\tilde{\theta}$ is asymptotically equivalent to

$\frac{1}{nI(\sigma)} =$

How would we make a confidence interval for σ using $\hat{\sigma}$ and this asymptotic variance?

Example: $X_1, \ldots, X_n$ iid with density $f_\theta(x) = (\theta+1)x^\theta$ for $0 \le \theta \le 1$.

Method of Moments

1. $E(X) = \int_0^1 (\theta+1)x^{\theta+1}dx$

2. $\theta =$

3. $\hat{\theta} =$

Method of Moments

1. Likelihood function: $L(\theta) =$

   log likelihood $l(\theta) =$

2. Differentiate with respect to θ

$$\frac{\partial log l(\theta)}{\partial\theta} =$$

3. Set to 0 and solve for $\tilde{\theta}$

Asymptotic Variance of $\tilde{\theta}$

$\frac{\partial log f_\theta(x)}{\partial \theta} =$

$\frac{\partial^2 log f_\theta(x)}{\partial \theta^2} =$

$I(\theta) = -E[\frac{\partial^2 log f_\theta(x)}{\partial \theta^2}] =$

The variance of $\tilde{\theta}$ is asymptotically equivalent to

$\frac{1}{nI(\theta)} =$