This lecture covered *Arrow’s Impossibility Theorem* and its generalization *Wilson’s Theorem*. Informally, this theorem states that if we have \( n \geq 3 \) voters trying to rank \( k \geq 3 \) alternatives, then the dictator function is the only way to aggregate each voter’s ranking to yield a complete ranking of the alternatives that satisfy:

- the relative positions of alternatives \( a \) and \( b \) depend only on each voter’s relative ranking of \( a \) and \( b \),
- if all voters prefer \( a \) to \( b \), then \( a \) must be above \( b \) in the final ranking.

To motivate the intuition behind Arrow’s theorem, we consider Condorcet’s Paradox, defined by the Marquis de Condorcet in 1785 in his *Essay on the Application of Analysis to the Probability of Majority Decisions* (the same essay that outlined his “Jury Theorem”).

**Example 1** (Condorcet’s Paradox). Consider the preferences of voters \( v_1, v_2, \) and \( v_3 \) for alternatives \( a, b, c \), where \( v_1 \) ranks \( a > b > c \), \( v_2 \) ranks \( b > c > a \), and \( v_3 \) ranks \( c > a > b \). A majority (2/3) of the voters rank \( a > b \), but similarly, 2/3 of the voters also rank \( b > c \) and \( c > a \), and thus there seems to be no rational ranking of \( a, b, c \) in a manner consistent with the voters’ preferences.

Before formally stating Arrow’s Impossibility Theorem, we will need some notation and definitions. For \( n \) voters ranking \( k \) alternatives, let the ranking submitted by voter \( i \) be denoted \( \sigma_i \in S(k) \), and let \( \sigma := (\sigma_1, \ldots, \sigma_n) \) denote the list of rankings given by the \( n \) voters.

We will now focus on the instance when \( k = 3 \), and we will denote the three alternatives as \( a, b, \) and \( c \). For ease of notation, we will map each voter’s ranking, \( \sigma_i \) to a tuple \( (x_i, y_i, z_i) \in \{-1,1\}^3 \), where \( x_i = 1 \) if \( \sigma_i(a) > \sigma_i(b) \), and \(-1\) otherwise. Similarly, \( y_i = 1 \) if \( \sigma_i(b) > \sigma_i(c) \) and \(-1\) otherwise, and \( z_i = 1 \) if \( \sigma_i(c) > \sigma_i(a) \), and \(-1\) otherwise. Finally, we will let \( x := (x_1, \ldots, x_n) \), \( y := (y_1, \ldots, y_n) \), and \( z := (z_1, \ldots, z_n) \).

**Remark 2.** Note that \( (x_i, y_i, z_i) \) corresponds to \( \sigma_i \) if, and only if, \( (x_i, y_i, z_i) \in \{-1,1\}^3 \setminus \{(1,1,1), (-1,-1,-1)\} \).

We now state the main definitions:

**Definition 3.** A constitution is a map \( F : S(3)^n \to \{-1,1\}^3 \).

The first coordinate of the image of \( F \) is the \( a \) vs. \( b \) outcome, the second is the \( b \) vs. \( c \) outcome and the third the \( c \) vs. \( a \) coordinate.

We now define three basic properties that, intuitively, are reasonable guidelines that we might hope “good” constitutions satisfy.
Definition 4. A constitution $F$ is Transitive if, for all sets of rankings $\sigma$, $F(\sigma) \in \{-1,1\}^3 \setminus \{(1,1,1),(-1,-1,-1)\}$; that is, $F$ is transitive if for all $\sigma$, $F(\sigma)$ is a proper ranking of the alternatives.

Definition 5. A constitution $F$ is Independent of Irrelevant Alternatives (IIA) if there exist functions $f,g,h : \{-1,1\}^n \rightarrow \{-1,1\}$ such that for all $\sigma$, we have $F(\sigma) = (f(x(\sigma)),g(y(\sigma)),h(z(\sigma)))$.

Definition 6. A constitution $F$ satisfies Unanimity if $\sigma_1 = \sigma_2 = \ldots = \sigma_n \Rightarrow F(\sigma) = \sigma_1$.

Example 7.

- The dictator function $F(\sigma) = \sigma_i$ clearly satisfies Transitivity, IIA, and Unanimity.

- The majority function $F(\sigma) = (\text{Maj}(x),\text{Maj}(y),\text{Maj}(z))$, where $\text{Maj}(v) = 1$ if $v$ contains at least as many 1s as $-1$s, satisfies IIA and Unanimity, but, as Condorcet’s Paradox demonstrates, does not satisfy Transitivity.

- The function $F(\sigma) = \tau$, where $\tau$ is the most frequently occurring permutation in $\sigma$ satisfies Transitivity, Unanimity, but not IIA, as can be seen by considering the outcomes corresponding to $\sigma = ((a,b,c),(a,b,c),(b,a,c),(b,c,a))$ and $\sigma = ((a,b,c),(a,c,b),(b,a,c),(b,a,c))$. $F(\sigma) = (1,1,-1) \neq (-1,1,-1) = F(\sigma')$, yet in both sets of rankings, $a > b$ for the first two voters, and $b < a$ for the second two voters, but the first coordinate of $F(\sigma)$ and $F(\sigma')$ differ, thus $F$ is not IIA.

We now state Arrow’s “Impossibility” Theorem:

Theorem 1 (Arrow’s “Impossibility” Theorem). Any constitution $F$ on $k \geq 3$ alternatives that is transitive, IIA, and satisfies unanimity is a dictator function; that is there exists some $i \in \{1,\ldots,n\}$ such that for all $\sigma$, $F(\sigma) = \sigma_i$.

The following definition will be helpful in our proof of Arrow’s theorem.

Definition 8. Voter $1$ is pivotal for $f : \{-1,1\}^n \rightarrow \{-1,1\}$ (denoted $I_1(f) > 0$) if there exist some $x_1,\ldots,x_n$ such that $f(-1,x_1,\ldots,x_n) \neq f(1,x_1,\ldots,x_n)$; we say voter $i$ is pivotal if the analogous statement holds.

Note that saying that voter $i$ is pivotal for $f$ exactly corresponds to saying that for the function $f$, the $i^{th}$ variable has nonzero influence.

The proof of Arrow’s theorem follows easily from the following lemma, due to Barbera;

Lemma 9 (Barbera ’82). Any IIA constitution $F = (f,g,h)$ on 3 alternatives that has $I_1(f) > 0$ and $I_2(g) > 0$ is non-transitive.

Proof. Since $I_1(f) > 0$ and $I_2(g) > 0$, there exist $x_2,\ldots,x_n$ and $y_1,y_3,y_4,\ldots,y_n$ such that $f(1,x_2,\ldots,x_n) \neq f(-1,x_2,\ldots,x_n)$ and $g(y_1,1,y_3,\ldots,y_n) \neq g(y_1,-1,y_3,\ldots,y_n)$.

Let $v = h(-y_1,-x_2,-x_3,\ldots,-x_n)$, and note that we can choose $x_1,y_2$ such that $f(x_1,x_2,\ldots,x_n) = g(y_1,y_2,\ldots,y_n) = v$. To conclude, note that the rankings given by $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$, and $z := (-y_1,-x_2,-x_3,\ldots,-x_n)$ are valid rankings, since for all $i$ $(x_1,y_i,z_i) \notin \{(-1,-1,-1),(1,1,1)\}$, yet $f(x) = g(y) = h(z)$, thus $F$ is not transitive.
Proof of Theorem 1: We first prove the theorem for the case that there are 3 alternatives. Since \( F \) is IIA, by assumption, without loss of generality let \( F = (f, g, h) \). Let \( I(f) := \{ \text{pivotal voters for } f \} \). Since \( F \) satisfies Unanimity, none of \( f, g, \) or \( h \) can be constant functions, and thus \( I(f), I(g), I(h), \) are all nonempty. Assume for the sake of contradiction that it is not the case that \( |I(f)| = 1 \) and \( I(f) = I(g) = I(h) \), then there is a pair of voters \( i \neq j \) which each are, respectively, pivotal for functions \( f, g \) or \( f, h \) or \( g, h \); by Barbera’s lemma \( F \) is not transitive, a contradiction, thus we conclude that \( F(\sigma) = G(\sigma_i) \), for some function \( G \). By the unanimity condition, \( G \) must be the identity function, so \( F(\sigma) = \sigma_i \), as desired.

We now prove the theorem for \( k > 3 \) alternatives. First observe that given a transitive IIA function \( F \) for \( k > 3 \) alternatives that satisfies unanimity, the restriction of \( F \) to any subset of just 3 alternatives will be a transitive IIA function on three alternatives that satisfies unanimity, and thus, from the above, we know that the restriction of \( F \) must be a dictator function. All that remains is to show that for all pairs of alternatives \((a, b), (a', b')\), for distinct \( a, b, a', b' \), the same dictator decides the relative position of \( a, b \) as \( a', b' \). To see this, let voter \( i \) be the dictator that decides the relative positions of \( a, b, a' \), and voter \( j \) the dictator that decides the relative positions of \( a, b, b' \), and note that \( i = j \), because the relative positions of \( a, b \) can not be decided by different dictators. \( \square \)

Given Arrow’s theorem, a natural direction is to relax our notions of a reasonable aggregation function. Along these lines, a natural question is:

What happens if we remove the unanimity constraint?

The first easy observation is that we only used the unanimity assumption in our proof of Arrow’s theorem in two places; concluding that \( I(f), I(g), I(h) \) are all nonempty, and in the final step where we say that since \( F(\sigma) = G(\sigma_i) \) it must be the case that \( G \) is the identity function. If, instead of assuming that \( F \) satisfies unanimity, we assume that for all pairs of alternatives \((a, b)\), there exists voter rankings \( \sigma, \sigma' \) such that \( a > b \) in \( F(\sigma) \) but \( a < b \) in \( F(\sigma') \), then we can still conclude that \( I(f), I(g), I(h) \) are nonempty, and the proof goes through to yield that \( F(\sigma) = G(\sigma_i) \) for some voter \( i \) and some function \( G \). What functions \( G \) can we use without violating the IIA condition?

Proposition 10. Given constitution \( F(\sigma) = G(\sigma_i) \) that is IIA and transitive, and for which for every pair of alternatives \((a, b)\), there exists some \( \sigma, \sigma_i \) for which \( G(\sigma_i) \) ranks \( a > b \) and \( G(\sigma') \) ranks \( a < b \), then either \( G(\sigma_i) = \sigma_i \), or \( G(\sigma_i) = -\sigma_i \), where \(-\sigma_i\) denotes the “reverse” of ranking \( \sigma_i \).

Proof. Note that \( G : \{-1, 1\}^{S(k)} \to \{-1, 1\}^{S(k)} \). Additionally, since \( F \), and thus \( G \) is transitive, we can write \( G = (g_1, \ldots, g_{|S(k)|}) \), where \( g_i : \{-1, 1\} \to \{-1, 1\} \). Note that the conditions of the proposition now imply that \( g_i \) can not be the constant function, and thus \( g_i = \pm Id \). Phrased such, the claim now amounts to showing that for all \( i, j \), \( g_i = g_j \) (ie either they are all the identity function, or all \(-1\) times the identity function). Assume for the sake of contradiction that there exist \( i, j \) s.t. \( g_i = -g_j \).

For clarity of notation, we replace the subscripts \( i, j \) by the pair of alternatives to which they refer, thus \( g_{a,b} \) indicates the relative ranking of \( a, b \). Thus we have two pairs \((a, b)\) and \((c, d)\) such that \( g_{a,b} = -g_{c,d} \). We now claim that we can find a triple \( r, s, t \) s.t. \( g_{r,s} = -g_{s,t} \). Indeed, consider \( g_{a,b} \) and \( g_{c,d} \), and assume without loss of generality that \( a \neq d \). If \( c = b \), then we have found such a triple. Otherwise, consider \( g_{a,b} \) and \( g_{b,d} \); if \( g_{a,b} = -g_{b,d} \), then we have found such a triple, otherwise we must have \( g_{b,d} = -g_{c,d} \).
To conclude, given \( g_{r,s} = -g_{s,t} \), without loss of generality assume that \( g_{r,s} = Id \) and that 
\( g_{s,t} = -Id \). We now consider the two cases that \( g_{r,t} = Id \) and \( g_{r,t} = -Id \). First, consider \( g_{r,t} = Id \): 
consider \( G((t > r > s)) \): in the result, it must be that \( r > s \), \( s > t \) and \( t > r \), which is not a valid ordering. Similarly, if \( g_{r,t} = -Id \): consider \( G((s > t > r)) \): in the result, it must be that \( s > r \), 
\( t > s \) and \( r > t \), which is not a valid ordering. 

We now characterize the set of constitutions that are IIA and transitive (and drop the condition 
that every pair \( a, b \) of alternatives can be ranked in both relative orderings \( a > b \) and \( b > a \)).

**Definition 11.** For a constitution \( F \), we write \( A >_F B \) if for all \( \sigma \) and all 
alternatives \( a \in A \) and \( b \in B \), it holds that \( F(\sigma) \) ranks \( a \) above \( b \).

**Theorem 2** (Wilson, ’72, Mossel ’10). A constitution \( F \) on \( k \) alternatives satisfies IIA and tran-
sitivity if, and only if there exists a partition of the alternatives into sets \( A_1, \ldots, A_s \) such that:

- \( A_1 >_F A_2 >_F \cdots >_F A_s \),
- If \( |A_r| > 2 \) then \( F \) restricted to \( A_r \) is a dictator on some voter \( j \), in that 
\( F^{A_r}(\sigma) = \pm \sigma^{A_r} \), where the superscript \( A_r \) denotes the restriction to the alternatives in \( A_r \).

Clearly any function of the above form is IIA and transitive, so it remains to prove that if \( F \) 
is IIA and transitive, then it has the claimed form. The following definitions will be helpful in our 
proof of Wilson’s theorem:

**Definition 12.** For a constitution \( F \), and two alternatives \( a, b \) write \( a >_F b \) if, for all \( \sigma \), 
\( F(\sigma) \) ranks \( a > b \). Write \( a \sim_F b \) if there exist \( \sigma, \sigma' \) such that \( F(\sigma) \) ranks \( a > b \) and \( F(\sigma') \) ranks \( a < b \).

**Lemma 13.** For a transitive and IIA function \( F \), if there exists two sets of voter rankings \( \sigma, \sigma' \) for 
which \( F(\sigma) \) ranks \( a > b \), and \( F(\sigma') \) ranks \( b > c \) then there exists a set of voter ranking \( \tau \) such that 
in \( F(\tau) \), \( a > c \).

_Proof._ Letting \( x, y \in \{-1, 1\}^n \) denote the vectors of relative preferences between \( a, b \) and \( b, c \), 
respectively, consider the voter rankings in which \( x = x_\sigma \), and \( y = y_{\sigma'} \). We can extend these 
preference lists into a set of valid rankings by setting the relative preferences between \( a, c \) to be 
\( z = -x \), and the preferences between all other pairs to be some arbitrary unanimous ranking. Thus 
we have constructed a set of voter preferences \( \tau \) which agrees with \( \sigma \) on the relative ranking \( a > b \) 
and agrees with \( \sigma' \) on the relative ranking \( b > c \).

**Corollary 14.** For a transitive, IIA function \( F \), the relations \( >_F \), and \( \sim_F \) are transitive. Addi-
tionally, if \( a >_F b \) and \( a \sim_F c \) and \( b \sim_F d \) then \( c >_F d \).

_Proof._ The transitivity of \( >_F \) is obvious, and the transitivity of \( \sim_F \) follows immediately from 
Lemma 13. If \( a >_F b \) and \( a \sim_F c \), then \( c >_F b \), since otherwise, given an instance \( \sigma \) for which 
\( F(\sigma) \) ranks \( c < b \), by Lemma 13 we can compose it with an instance for which \( a < c \), yielding an instance 
in which \( a < b \), contradiction \( a >_F b \). Thus if \( a >_F b \), \( a \sim_F c \), \( b \sim_F d \) then \( c >_F b \). Applying 
the same argument to \( c >_F b \) and \( b \sim_F d \) yields that \( c >_F d \), as desired.

The proof of Wilson’s theorem now follows easily from the above corollary and Arrow’s theorem.

_Proof of Theorem 2:_ We first leverage Corollary 14 to show that there exists a partition of the 
alternatives into sets \( A_1 >_F A_2 >_F \cdots >_F A_s \). Indeed, for a given alternative \( a \), let \( A := \{ b :
For \( a' \not\in nA \), define the set \( A' \) analogously, and note that since \( a' \not\in A \), without loss of generality we may assume that \( a <_F a' \). By Corollary 14, \( A <_F A' \), and thus the construction of the partitions is well-defined. To conclude, we note that for every pair of alternatives \( a, b \) that lie in the same partition, from the definition of \( \sim_F \), there exist outcomes for which \( F \) ranks \( a > b \) and \( b > a \), and thus we may apply Arrow’s theorem to the restriction of \( F \) to each partition, yielding that the restriction of \( F \) to a partition \( A_i \) is a dictator function \( G(\sigma_i) \), and thus by Proposition 10, the restriction of \( F \) to any partition is either the dictator function \( \sigma_i \) or \( -\sigma_i \), as claimed. □

We concluded with a final remark that if voters don’t need to provide strict ordering, and instead can indicate ties, then one-sided versions of Arrow’s and Wilson’s theorem hold—though considering such a general settings seems to only obfuscate the interesting characterizations.