Fall 2010 : Social Choice and Networks
CS 294 (063) / Econ 207 A / Math C223A / Stat C206A
Topic : Fair Allocations

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1 Overview

Today's lecture is about "fair" allocation of resources to agents. Several notions of fairness could be defined. For example one could maximize the "social welfare" which is the sum of "happiness" of each agent, or the "happiness" of the most "unhappy" person, or proportionalness where every agent gets at least 1/k of the total according to their own measure. We focus on the envy freeness which means every agent prefers his own allocation over anyone else's. Although envy free partitions are always possible when the goods are divisible, it is not the case for indivisible goods. As an example consider the case where there is only one good and many agents.

Empirically the notion of fair allocations was familiar to the ancient egyptians who wanted to partition their land such that each would receive the same amount of irrigation. Such partitions are also known as "nile-partitions". Mathematically this problem has been studied by [Steinhaus, Banach, Knaster 48]. Others [3, 4] have looked at it from a game theoretic perspective where the goal is a protocoal for a group of agents to arrive to a "fair" allocation. As an example, consider the case where two agents are trying to allocate a cake. In this case a protocol for achieving a fair allocation would be to let one cut the cake the other choose.

In today's lecture we focus on the computational complexity of this fair allocations which minimize envy.

2 Discrete Envy Free Allocations

This setting consists of k agents and m indivisible goods. Also let u_{ij} be the utility of good i for agent j. Let u_{iC} be the utility of agent i for a subset C of goods. We consider the case where the utility is additive, i.e. $u_{iC} = \sum_{j \in C} u_{ij}$. Also denote a partition (allocation) as the $C_1, C_2, \ldots, C_K$ where C_j denotes the set of goods allocated to the agent j.

The envy e_{pq}, denotes how much agent p envies agent q's allocation and can be defined in terms of the utility functions as :

$$e_{pq} = \max(0, u_{p,C_q} - u_{p,C_p}) \tag{1}$$

The envy of an allocation A is $e(A) = \max_{p,q} e_{pq}$.

It turns out finding the allocation that minimizes envy is NP hard. We instead present polynomial time algorithms which have upper bounds on the envy.

2.1 A bound in terms of atomicity

A useful parameter of the allocation problem is $\alpha := \max_{p,j} u_{p,j}$ which is the maximum utility of any good by any person.

Theorem [DallAglio - Hill 03][1]: There exists an allocation A with $e(A) \leq \alpha(2k)^{3/2}$.

Theorem [Liption *et al.*][2] There is an polynomial time algorithm to compute an allocation A such that $e(A) \leq \alpha$.

Proof Sketch: The proof is an algorithm for achieving an allocation with an envy utmost α. Consider the the envy-graph of an allocation which is a directed graph with an edge $i \to j$ if i envies j. The algorithm is to repeat the following steps till all the goods are allocated :

1. Find and eliminate all the directed cycles from the envy-graph.
2. Give the next good to an agent that no-one envies (any node with in-degree = 0).

The idea is that if a cycle exits in the envy graph one can eliminate the cycle by rotating the allocations. One has to prove that this step does not introduce new cycles and the process terminates. If B is the graph obtained from A by eliminating the cycles. We have the following claims :

1. $e(B) \leq e(A)$
2. envy-graph of B is acyclic ($\exists i$ with in-degree = 0).

The fact that $e(B) \leq e(A)$ is easy to see because removing any cycle only reduces the envy of any agent. To see that the process terminates consider the overall utility $\sum_i u_{iC_i}$ (also called "social welfare"). This quantity strictly goes up every time a cycle is eliminated. Since this quantity is bounded, the process has to terminate after finite steps. Giving the next good to the person that no one envies does not increase the envy of the allocation either. Therefore one can achieve an envy of α using this procedure.

To obtain bound on the number of steps taken by the algorithm we can analyze the envy-graph. Notice that removing a cycle in the graph reduces the number of edges of the envy graph by at least one. The number of edges from outside the cycle to nodes in the cycle remain the same as the the allocations are just shifted by one, and is equivalent to re-labeling the nodes. The number edges from nodes in the cycle to outside may go down as the utility of the nodes go up which may cause some nodes to no longer envy others. The number of edges in the cycle goes down by at least 1 after the rotation. Since there are $O(k^2)$ edges, this process terminates in at most $O(k^2)$ steps. One can find a cycle in the graph in $O(k)$ steps using a breath first search and the overall algorithm terminates in most $O(mk^3)$ steps.

3 Continuous Fair Allocations

In this setting for each partition $A = A_1, A_2, \ldots, A_k$, each agent has a utility $\mu_i(A_j)$ according to some measure μ_i. An allocation is fair if $\mu_i(A_i) \geq \mu_i(A_j)$. Its easy to see that a fair allocation achieves $\mu_i(A_i) \geq 1/k$.

Let $A = A_1, A_2, \ldots, A_k$ be a partition and $\mu_1, \mu_2, \ldots, \mu_k$ be measures. Denote $M(\mu, A)$ for the matrix $M_{ij} = \mu_i(A_j)$.

Theorem [Dubins and Spanier] [3] (proved using a theorem from [5]): For all non-atomic $\mu_1, \mu_2, \ldots, \mu_k$ on the same space the set $M(u, A)$ where A runs over all partitions is compact and convex.

One can partition A into ϵ small chunks, such that no chunk has utility greater than ϵ and use the algorithm we proposed in the discrete case to to achieve an allocation with envy at most ϵ. This with the compactness result implies that existence of fair allocations for the continuous case.

References

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[2] R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. On approximately fair allocations of indivisible goods. In Proceedings of the 5th ACM conference on Electronic commerce (EC '04). ACM, New York, NY, USA, 125-131.

[3] Dubins, L. E. and E. H. Spanier. 1961. How to cut a cake fairly. American Mathematical Monthly 68(1):117.

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