Errors in Binary Voting

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Draft
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Unbiased Signals

- In the next few weeks we will deal with the case of unbiased signals / uniformed voters.
- Why do we do it?
Unbiased Signals

• In the next few weeks we will deal with the case of unbiased signals / uniformed voters.
• Why do we do it?
• These measures provide “stress-test” for the voting methods we are using:
  • If voters all have strong correlated opinions then:
    • Small effects of (small) errors in the voting scheme
    • Will not see irrational outcomes.
  • Outcome hard to manipulate.
Unbiased Signals

• What are unbiased signals / uniformed voters?
Unbiased Signals

• What are unbiased signals / uniformed voters?

• Worst case scenarios: exists voting configurations resulting in errors/manipulation/etc.

• Average case scenarios: On average there is a good probability of errors/manipulation/etc.

• Average with respect to the most uniformed measure = the uniform measure.
Definition of voting schemes

• Today topic is errors of voting schemes on binary decisions.
• A population of size \( n \) is to choose between two options / candidates.
• A voting scheme is a function that associates to each configuration of votes which option to choose.
• Formally, a voting scheme is a function \( f : \{-1,1\}^n \rightarrow \{-1,1\} \).
• Two prime examples:
  - Majority vote,
  - Electoral college.
Properties of voting schemes

- Some properties of voting schemes:
- We will always assume that candidates are treated equally:
  The function $f$ is fair: if $f(-x) = -f(x)$.
- We always assume that stronger support in a candidate shouldn’t heart her:
  The function $f$ is monotone: $x \geq y \Rightarrow f(x) \geq f(y)$, where $x \geq y$ if $x_i \geq y_i$ for all $i$.
- Note that both majority and the electoral college are anti-symmetric and monotone.
Democracy and voting schemes

- Two interpretations of democracy:
  - "Weak democracy" - each voter has the same power: There exists a transitive group $\Gamma \subset S_n$ such that for all $\sigma \in \Gamma$ and all $x$ it holds that
    \[ f((x_{\sigma(i)})) = f((x_i)) \] (***)
  - "Strong democracy" - each set of voters has the same power - (*** holds for all $\sigma \in S_n$).

- Easy: Monotonicity + fairness + strong democracy $\Rightarrow f = \text{majority}$.

- But: Electoral college is weak democracy (mathematically)
Errors in voting

- **Claim:** Any non-constant voting scheme is prone to errors.
- **Pf:** Since it is not constant there exist x and y such that $f(x) \neq f(y)$.

- **Claim:** Any non-constant voting scheme is prone to an error of a single voter.

- **Pf:** Otherwise whenever we change a single coordinate the value of $f$ stays the same. But this means that any # of coordinates changes does not change the value of f. .
To the **uniform measure**

- Assume $x$ is chosen *uniformly* in $\{-1,1\}^n$.
- Let $y = N_\varepsilon(x)$ is obtained from $x$ by flipping each of $x$ coordinates with probability $\varepsilon$.
- **Question:** What is the probability that the population voted for who they meant to vote for?
- What is $S_f(\varepsilon) = P[f(x) = f(y)]$?
- Which is the most *sensitive / stable* $f$?
- Is there an $f$ which is both stable and sensible?
- **Maj? Electoral college?**
Stability without democracy

• $N(x, y) = P[N_\varepsilon(x) = y]$, $\eta = 1 - 2\varepsilon$
  and $Z(f, \eta) := \langle f, Nf \rangle = E[f(x) f(y)]$
• $S(f,\varepsilon) = (Z(f,\eta)+1)/2$
• $N$ has the eigenvectors $u_S(x) = \prod_{i \in S} x_i$, corresponding to the eigenvalues $\eta^{|S|}$.
• $P[f(x) = f(N_\varepsilon(x))] = \frac{1}{2} + E[f(x)f(N_\varepsilon(x))]/2 = \frac{1}{2} + \langle f, Nf \rangle/2$.
• Write $f(x) = \sum_S f_S u_S(x)$.
• Since $\langle f, 1 \rangle = 0$, $\langle f, Nf \rangle = \sum_{S \neq \emptyset} f_S^2 \eta^{|S|} \leq \eta$
  and therefore $S_f(\varepsilon) = 1 - \varepsilon$.
• Dictatorship, $f(x) = x_i$ is the only optimal function.
Stability with democracy

- How stable can we get with democracy?
- Thm Sheffield (1899):
  \[ \lim_{n \to \infty} Z(f, \eta) = \frac{\arcsin(\eta)}{\pi} \text{ for } f = \text{maj}_n. \]
- When \( \varepsilon \) is small \( S_f(\varepsilon) \sim 1 - 2 \varepsilon^{1/2}/\pi. \)
- Pf?
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Stability with democracy

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• Pf: Let \( N = n^{-1/2} \sum x_i, M = n^{-1/2} \sum y_i \)
• By CLT \( E[f(x) f(y)] \to E[\text{sgn}(N) \text{sgn}(M)] = \)
  \[ = 1 - 2 P[\text{sgn}(N) \neq \text{sgn}(M)] = \]
  \[ = 1 - P[N > 0, M < 0]. \]
• Write \( M = a N - b U \) where \( a = \eta \) and \( a^2 + b^2 = 1 \) and \( N \) are independent. Then
• \( P[N > 0, M < 0] = P[0 < N < (b/a)U] = \arctan(b/a)/2 \pi \)
• Why?
• This give the result using trigonometric identities.
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- How stable can we get with democracy?
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  - \( \lim_{n \to \infty} Z(f, \eta) = \frac{\arcsin(\eta)}{\pi} \) for \( f = \text{maj}_n \).
  - When \( \varepsilon \) is small \( S_f(\varepsilon) \sim 2 \varepsilon^{1/2}/\pi \).
  - Claim: An \( n^{1/2} \times n^{1/2} \) electoral college gives \( S_f(\varepsilon) = \Theta(\varepsilon^{1/4}) \).
- Why?
Stability with democracy

• How stable can we get with democracy?

• Thm Sheffield (1899):
  \[ \lim_{n \to \infty} Z(f, \eta) = \frac{\arcsin(\eta)}{\pi} \text{ for } f = \text{maj}_n. \]

• When \( \varepsilon \) is small \( S_f(\varepsilon) \approx 2 \varepsilon^{1/2}/\pi. \)

• Thm (Majority is Stablest; M-O’Donnell-Olseskiwsz):
  If \( f = f_n \) satisfies
  \[ \max \{ e_i(f) : 1 \leq i \leq n \} = o(1), \text{ then} \]
  \[ \lim_{n \to \infty} S_f(\varepsilon) \geq \frac{1}{2} - \frac{\arcsin(1 - 2\varepsilon)}{\pi}. \]

  \( \Rightarrow \) most stable “weak democracy” = maj.

• Won’t do proof. Can hear a bit about it tomorrow at 4pm.
Majority is Least Stable

• Interestingly if we are just interested in a single error then:

• **Thm:**
  • Among all monotone functions, majority maximizes the probability $P[f(x) \neq f(y)]$ where $y$ is obtained from $x$ by a random flip of one bit.

• **Pf:** We want to maximize:

  • $\sum \{f(y) - f(x) : y \text{ directly above } x\} =$
  • $\sum_{k=0}^{n} \sum \{(k - (n-k))f(x) : x \text{ s.t. } \#(1,x) = k\} =$
  • $\sum_{k=0}^{n} \sum \{(n-2k)f(x) : x \text{ s.t. } \#(1,x) = k\}.$

• So Majority is least stable for fixed flip probability and most stable for flip probability $\ll 1/n.$
Getting **sensitive**

- **How sensitive** can a fair monotone functions be?
- Interesting in **learning, neural networks, hardness amplification** ...
- \( \text{maj}_n \) maximizes the isoperimetric edge bounds among all monotone functions and \( I(\text{maj}_n)^2 \sim 2n/\pi \).
- By Russo's formula: \( I(f) = Z'(f,1) \).
- But \( Z'(f,1) = \sum_S |S| f_S^2 \).
- Consider the following relaxation of the problem:
  - minimize \( \sum_S a_S \eta^{|S|} \) under the constraints:
    - \( \sum_S a_S = 1, a_S \geq 0, \sum_S |S| a_S \sim \leq (2n/\pi)^{1/2} := \alpha \)
    - We get that \( Z(f,\eta) \sim \geq \eta^\alpha \)
    - In particular, any monotone function requires randomly flipping at least order \( n^{1/2} \) votes to have probability \( > 0.00001 \) to flip the elections.
**Getting sensitive**

- **Kalai:** Are there any functions that are so sensitive?
- **Kalai:** Is it enough to flip $n^{1/2}$ of the votes in order to flip outcome with probability $\Omega(1)$?
- **Thm [M-O’Donnell]:**
  - rec-maj-$k$ satisfies $\langle f, N_f \rangle \sim \eta^{\alpha(n,k)}$ where $\alpha(n,k) \sim n^{\beta(k)}$ and $\beta(k) \to \frac{1}{2}$ as $k \to \infty$ (enough to flip $n^{1-\beta(k)}$)
  - rec-maj with increasing arities gives that it is enough to flip $\log^+(n) n^{1/2}$ where $t = \frac{1}{2} \log_2(\pi/2)$.
  - Talgrand’s random function gives that it is enough to flip $c n^{1/2}$. 

![Diagram](image-url)